LINEAR ORDINARY D.E.

(1) $a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ — general form

(2) $\frac{dy}{dx} + P(x)y = f(x)$ — standard form

$$P(x) = \frac{a_0(x)}{a_1(x)}$$

$$f(x) = \frac{g(x)}{a_1(x)}$$

The solution of (2) is the sum of two solutions: $y = y_p + y_c$; where

y_c — is the solution of the associated homogeneous diff. eq.

$$\boxed{\frac{dy}{dx} + P(x)y = 0} \quad (3)$$

and y_p — is a particular solution of the nonhomogeneous equation (2)

y_c — found by solving a SEP. VAR. D.E (3)

y_p — found by using variation of parameters method

A solution method for solving 1st order linear ODE (2) is the application of the 3-step integration factor method.

1) Define $h(x) = \int P(x)\,dx$ $\Longrightarrow P(x) = h'(x)$

2) Multiply both side of (2) with e^h

$$\left(\frac{dy}{dx} + P(x)\,y\right)e^h = f(x)\cdot e^h$$

$$y'e^h + P(x)\cdot e^h\cdot y = f(x)\cdot e^h$$

$$\underbrace{y'e^h + h'e^h y}_{d(y\cdot e^h)} = f(x)\cdot e^h$$

$$d\left(y\cdot e^h\right) = f(x)\,e^h \quad (3)$$

3) integrate both side of (3)

$$\int \frac{d}{dx}\left(y\cdot e^h\right)dx = \int f(x)\,e^h\,dx$$

$$\int \frac{d}{dx}(y \cdot e^{h})\, dx = \int f(x) e^{h} dx$$

$$y \cdot e^{h} = \int f(x) e^{h} dx + c \quad ; \; h(x) = \int P(x)\, dx$$

$$y = e^{-h}\left(\int f(x) e^{h} dx + c\right)$$

$$\boxed{y(x) = e^{-\int P(x)\, dx}\left(\int f(x)\, e^{\int P(x)\, dx} dx + c\right)}$$

Example $\quad \boxed{y' + \underbrace{y \cdot \tan x}_{P(x)} = \underbrace{\sin 2x}_{f(x)}}$

1) $h(x) = \int P(x)\, dx = \int \tan x\, dx = \int \frac{\sin x\, dx}{\cos x}$

$$= -\int \frac{(-\sin x)}{\cos x} dx = -\int \frac{(\cos x)'}{\cos x} dx$$

$$= -\ln|\cos(x)| = \ln|\sec(x)|$$

2) multiply the D.E. w/ $e^{h(x)}$

$$e^{h(x)} = e^{\ln|\sec(x)|} = \sec(x)$$

$$e^{-h(x)} = (\sec(x))^{-1} = \cos(x)$$

$$f(x)\cdot e^{h(x)} = \sin 2x \cdot \sec(x) = 2\sin x \cos x \cdot \frac{1}{\cos x}$$

$$= 2\sin x$$

$$y = y(x) = e^{-\int P(x)\,dx}\left(\int f(x)\, e^{\int P(x)\,dx}\,dx + c\right) =$$

$$= \cos x \left[\int 2\sin x\,dx + c\right] = \cos x\left[-2\cos x + c\right]$$

$$y = \underbrace{c\cos x}_{y_c = c\cos x} \underbrace{-2\cos^2 x}_{y_p = -2\cos^2 x}$$

$$y = y_c + y_p$$

verify y_c satisfies the assoc. homogeneous D.E.: $y' + y\tan x = 0$

$$\boxed{y' + y \cdot \tan x = 0}$$; $y = y_c = c \cos x$

$$y_c' + y_c \cdot \tan x = -c \sin x + c \cos x \cdot \frac{\sin x}{\cos x}$$

$$= -c \sin x + c \sin x = 0$$

verify y_p is a particular solution for the given D.E. $\quad y_p = -2\cos^2 x$

$$y_p' + (\tan x)\, y_p = \sin 2x$$

$$y_p' = (-2)\left[(-1)\sin x\right] 2 \cdot \cos x = \underbrace{2 \cdot 2 \sin x \cos x}_{\sin 2x} =$$

$$y_p' = 2\sin 2x$$

$$y_p' + y_p \tan x = 2\sin 2x - 2\cos^2 x \cdot \frac{\sin x}{\cancel{\cos x}} =$$

$$= 2\sin 2x - \sin 2x = \sin 2x$$

$$y' + y\tan x = \sin 2x \quad / \; e^{\int \tan x} \qquad ; \; e^{\int \tan x} = \frac{1}{\cos x}$$

$$\frac{dy}{dx} \cdot \frac{1}{\cos x} + y \tan x \frac{1}{\cos x} = \sin 2x \cdot \frac{1}{\cos x}$$

$$\frac{dy}{dx} \frac{1}{\cos x} + y \frac{\sin x}{\cos^2 x} = 2\sin x \cancel{\cos x} \frac{1}{\cancel{\cos x}}$$

$$\left(y \cdot \frac{1}{\cos x}\right)' = 2\sin x \quad \Rightarrow \int \left(y \frac{1}{\cos x}\right)' = \int 2\sin x \, dx$$

$$y = \cos x \left[\int 2\sin x \, dx + c\right] \Rightarrow \boxed{y = c \cdot \cos x - 2\cos^2 x}$$

Example:

$$y' = (x+1)e^{-x}y^2 \iff \frac{1}{y^2}y' = (x+1)e^{-x} \quad ; y' = \frac{dy}{dx}$$

$$\int \frac{1}{y^2}\,dy = \int (x+1)e^{-x}\,dx$$

$$-\frac{1}{y} = \underbrace{\int (x+1)e^{-x}\,dx}_{I}$$

$$I = \int (x+1)e^{-x}\,dx = \int (x+1)\left(-e^{-x}\right)'dx = (x+1)\left(-e^{-x}\right) -$$

$$-\int (x+1)'\left(-e^{-x}\right)dx = -(x+1)e^{-x} + \int e^{-x} =$$

$$= -(x+1)e^{-x} - e^{-x} + c$$

$$-\frac{1}{y} = -(x+1)e^{-x} - e^{-x} + c = -e^{-x}(x+2) + c$$

$$y = \frac{1}{e^{-x}(x+2) - c}$$