SEPARABLE ODE

$\frac{dy}{dx} = f(x,y)$; assume $f(x,y) = g(x)$

$$\frac{dy}{dx} = g(x) \Rightarrow dy = g(x)\,dx \Rightarrow \int dy = \int g(x)dx$$

antiderivative of $g(x)$

$y = G(x) + c \rightarrow$ arbitrary real-value

Ex: $\frac{dy}{dx} = \underbrace{3 + e^{-3x}}_{g(x)}$

$$y = \int g(x)\,dx = \int (3 + e^{-3x})\,dx = \int 3\,dx + \int e^{-3x}dx$$

$$= (3x + c_1) - \frac{1}{3}e^{-3x} + c_2 = 3x - \frac{1}{3}e^{-3x} + (c_1 + c_2)$$

$$y = 3x - \frac{1}{3}e^{-3x} + c$$

General solution method for SEP. VAR. D.E.

$$\boxed{\frac{dy}{dx} = g(x)\cdot h(y)} \qquad \frac{1}{h(y)}\cdot\frac{dy}{dx} = g(x) \quad (1)$$

$p(y) = \frac{1}{h(y)}$, where $y = \phi(x)$ (2)

use (1)+(2) : $p(\phi(x))\cdot \phi'(x) = g(x) \Longleftrightarrow$

$$\int p(\phi(x))\cdot \phi'(x) = \int g(x) \iff \int \frac{1}{h(y)}\,dy = \int g(x)\,dx$$

$$H(y) = G(x) + c$$

Ex. (3) $(1+x)\,dy - 3y\,dx = 0$; find $y = \phi(x)$?

$$\frac{(1+x)\,dy}{(1+x)y} - \frac{3y\,dx}{(1+x)\,y} = 0 \iff \frac{dy}{y} - \frac{3}{1+x}\,dx = 0$$

$$\underbrace{\frac{1}{y}\,dy = \frac{3}{1+x}\,dx}_{\text{apply } \int .} \quad ; \quad \boxed{\frac{dy}{dx} = \frac{3y}{1+x}}$$

$$\int \frac{1}{y}\,dy = \int \frac{3}{1+x}\,dx \quad ; \quad \ln|y| = 3\ln|1+x| + \ln c$$

$$\ln|y| = \ln|(1+x)^3| + \ln c \Rightarrow \ln|y| = \ln|c(1+x)^3|$$

$$\boxed{y = c(1+x)^3}$$ solution of (3)

Ex: vehicles – acceleration rate is not constant over time; as speed increases acceleration rate decreases

max. velocity => acceleration is zero

$$a = \frac{du}{dt} = \dot{u}(t)$$

u - velocity

a – rate of change of speed.

t – time

$$\boxed{\frac{du}{dt} = \alpha - \beta \cdot u}$$

; α, β parameters related vehicle/engine performance

$$\frac{du}{dt} = \underbrace{f(t)}_{1} \cdot \underbrace{h(u)}_{\alpha - \beta u}$$

; $\frac{du}{dt} = \alpha - \beta u \Leftrightarrow \frac{du}{\alpha - \beta u} = dt$

$$\int \frac{1}{\alpha - \beta u}\, du = \int dt$$

; $\boxed{-\frac{1}{\beta} \ln(\alpha - \beta u) = t + c}$ (4) $u \neq \dot{u}(t)$

initial value: at $t = 0$; $u = u_0$ => use in (4)

$$-\frac{1}{\beta} \ln(\alpha - \beta u_0) = 0 + c \Rightarrow c = -\frac{1}{\beta} \ln(\alpha - \beta u_0) \quad (5)$$

use (5) in (4) => $-\frac{1}{\beta} \ln(\alpha - \beta u) = t - \frac{1}{\beta} \ln(\alpha - \beta u_0)$

$$\ln\left(\frac{\alpha - \beta u_0}{\alpha - \beta u}\right) = \beta t \Rightarrow t = \frac{1}{\beta} \ln\left(\frac{\alpha - \beta u_0}{\alpha - \beta u}\right) \Rightarrow \boxed{u = \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + u_0 e^{-\beta t}}$$