Solution of an Ordinary Differential Equation (ODE)

Any function ø, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an n-th order ODE reduces the equation to an identity, is said to be a solution of the equation on the interval.

(1) $\underbrace{x}_{a_1(x)} y' + \underbrace{y}_{a_0(x) = 1} = \underbrace{0}_{g(x)}$; $y = y(x)$

solution of (1)

$\boxed{y(x) = \frac{1}{x}}$

$y'(x) = x^{-1} = -x^{-1-1} = -\frac{1}{x^2}$

$x \cdot \left(-\frac{1}{x^2}\right) + \frac{1}{x} = -\frac{1}{x} + \frac{1}{x} = 0$

$I = (-\infty, 0)$ OR $I = (0, \infty)$ because

$y, y', \ldots$ have to be continuous on I

I cannot be $(-\infty, 0) \cup (0, \infty)$

Implicit vs. Explicit Solution of an ODE.

A relation $G(x,y) = 0$ is said to be an implicit solution of an ODE on an interval I if there is at least one function ϕ that satisfies both, the relation and the ODE on the interval.

$G(x,y) = x^2+y^2 = 25$ is an implicit solution of eq: $\frac{dy}{dx} = -\frac{x}{y}$ ①

$x^2+y^2=25$; $y^2 = 25-x^2$; $y_{1,2} = \pm\sqrt{25-x^2}$

$y_1 = \sqrt{25-x^2} = (25-x^2)^{1/2}$ $I = (-5,5)$

$$\frac{dy_1}{dx} = -\frac{2x}{2\sqrt{25-x^2}} = -\frac{x}{\sqrt{25-x^2}} = -\frac{x}{y_1}$$

y_1

$y_1 = \sqrt{25-x^2}$

$y_2 = -\sqrt{25-x^2}$

$y_1, y_2 =$ explicit sol. of D.E. ①

$I = (-5,5)$

$G(x,y) =$ implicit solution

Family of solutions for ODE

A solution containing an arbitrary constant represents a set $G(x,y,c)=0$ of solutions called a one-parameter family of solutions

An n^{th} order D.E. $F(x,y,y',\dots y^{(n)})=0$ may have an n-parameter family of solutions of the form $G(x,y,c_1,c_2,\dots c_n)=0$

A solution without any arbitrary parameters is called a particular solution

Ex: $y=\left(\frac{x^2}{4}+c\right)^2$ a one parameter family of solutions for: $y'=x\sqrt{y}$

$x\in I,\ I=(-\infty,\infty)$

Q: is $\boxed{y=0}$ a solution for $y'=x\sqrt{y}$?

trivial solution is a particular sol.

INITIAL VALUE PROBLEMS

We seek solutions $y(x)$ for DE such that $y(x)$ satisfies predetermined conditions imposed on the dependent variable and its derivatives

Given I, $x_0 \in I$ solve for:

$$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$ — normal form of ODE

subject to: $y(x_0) = y_0$, $y'(x_0) = y_1$, $\vdots$, $y^{(n-1)}(x_0) = y_{n-1}$ — initial conditions

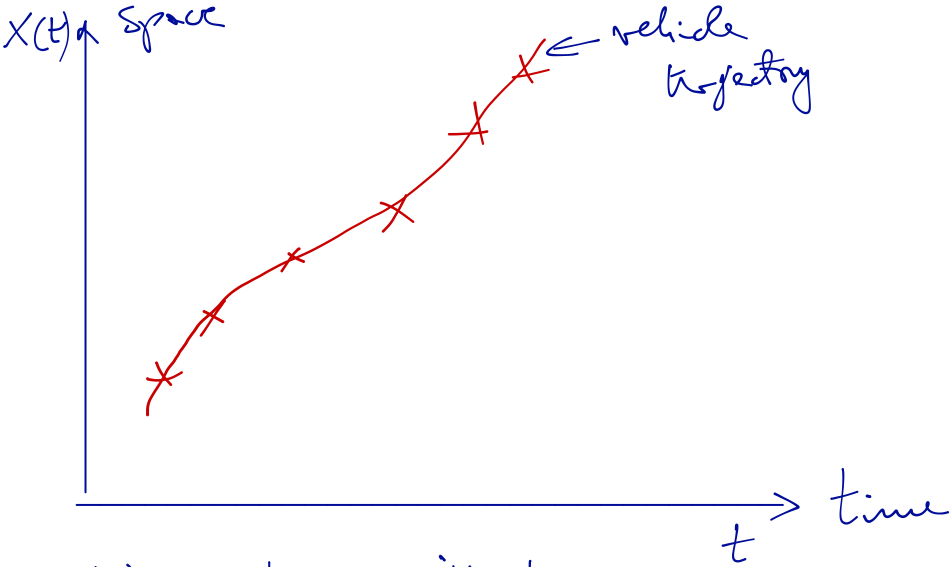
$y_0, y_1 \dots y_{n-1} \in R$

1st order ODE

IVP $\begin{cases} \frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$

2nd order ODE

IVP $\begin{cases} \frac{d^2y}{dx^2} = f(x, y, y') \\ y(x_0) = y_0 \\ y'(x_0) = y_1 \end{cases}$



$X(t)$; t iV: $\begin{cases} X(t_0) = X_0 \\ \dot{X}(t_0) = V_0 \end{cases}$

$$\ddot{X}(t) = f(t, X(t), \dot{X}(t))$$ – acceleration

Ex: given: $y' + 2xy^2 = 0$ → general solution is $y(x) = \frac{1}{x^2+c}$; if $y(0) = -1$ (iVP) then ⇒

⇒ $c = -1$; ⇒ solution is unique $\boxed{y(x) = \frac{1}{x^2-1}}$

Ex: (non-unique sol. of iVP)

let $\frac{dy}{dx} = xy^{1/2}$; $y(0) = 0$; $y(x) = \frac{x^4}{16}$

and $y(x) = 0$

are 2 distinct solutions

A first order IVP has a unique solution if given R, a rectangular region in the xy plane defined by $a \le x \le b$, $c \le y \le d$ containing (x_0, y_0) and if $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on R.

There exist an interval $I_0: (x_0 - h, x_0 + h)$, $h > 0$ contained in $[a,b]$, on which a unique solution can be defined for the given IVP

Ex: $\frac{dy}{dx} = xy^{1/2}$; $f(x,y) = xy^{1/2}$

$$\frac{\partial f}{\partial x} = \frac{1}{2} x y^{-1/2} = \frac{1}{2} \frac{x}{y^{1/2}}$$; not continuous in $y=0$!

Ex. 2^{nd} order ODE IVP

$$x''(t) + 16x(t) = 0 \quad \& \quad \text{I.V.} \begin{cases} x\left(\frac{\pi}{2}\right) = -2 \\ x'\left(\pi/2\right) = 1 \end{cases}$$

given the solution: $\boxed{x(t) = c_1 x_1(t) + c_2 x_2(t)}$, a 2-parameter family of solutions

where $x_1(t) = \cos 4t$ & $x_2(t) = \sin 4t$

use 1^{st} IV $x\left(\frac{\pi}{2}\right) = -2$; $c_1 \cos 4\frac{\pi}{2} + c_2 \sin \frac{4\pi}{2} = -2 \Rightarrow$

$c_1 + c_2 \cdot 0 = -2 \quad \Rightarrow \quad \boxed{c_1 = -2}$

use 2nd i.v.: $x'(\pi/2) = 1$; $-4c_1 \sin(4\,\pi/2) + 4c_2 \cos\left(\frac{4\pi}{2}\right) = 1$

$-4(-2) \cdot 0 + 4 \cdot c_2 = 1 \quad \Rightarrow \quad c_2 = 1/4$

solution of the IVP: $x(t) = -2\cos 4t + \frac{1}{4}\sin 4t$