(1) $2xy\,dx + (x^2-y^2)\,dy = 0$ (*)

$M_y = 2x = N_x$, hence (*) is exact.

Let $u(x,y) = \int 2xy\,dx + T(y)$. Then

$$u(x,y) = x^2 y + T(y).$$

$x^2 - y^2 = u_y = x^2 + T'(y)$. Hence $T'(y) = -y^2$ and therefore

$T(y) = -\frac{1}{3}y^3 + c_1$, c_1 - constant.

Therefore, the general solution is

$$\boxed{x^2 y - \frac{1}{3}y^3 = C.}$$

(2) $e^{-y}dx - (2y + xe^{-y})dy = 0$ (*)

$M_y = -e^{-y} = N_x$, hence (*) is exact.

Let $u(x,y) = \int e^{-y}dx + T(y)$. Then

$$u(x,y) = xe^{-y} + T(y).$$

$-2y - xe^{-y} = u_y = -xe^{-y} + T'(y) \Rightarrow T'(y) = -2y.$

Thus, $T(y) = -y^2 + c_1$, c_1 - constant.

Therefore, the general solution is

$$\boxed{xe^{-y} - y^2 = C}$$

(3) $(1+y^2\sin 2x)dx - 2y\cos^2 x\,dy = 0.$ $(*)$

$$M_y = 2y\sin 2x \qquad N_x = -2y\cdot 2\cos x\cdot(-\sin x) =$$
$$= 4y\sin x\cos x = 2y\cdot 2\sin x\cos x =$$
$$= 2y\sin 2x.$$

We see, that $M_y = N_x$, hence $(*)$ is exact.

Hence we find $u = u(x,y)$ in the form

$$u(x,y) = \int -2y\cos^2 x\,dy + T(x) = -y^2\cos^2 x + T(x).$$

$$1+y^2\cdot\sin 2x = u_x = -y^2\cdot 2\cos x\cdot(-\sin x) + T'(x)$$
$$1+y^2\cdot\sin 2x = y^2\cdot\sin(2x) + T'(x).$$

Thus, $T'(x) = 1 \Rightarrow T(x) = x + c_1$, c_1 - constant.

Therefore, $u(x,y) = -y^2\cos^2 x + x$ and the general solution is $\boxed{-y^2\cos^2 x + x = C.}$

(4) $3x^2(1+\ln y)dx = \left(2y - \frac{x^3}{y}\right)dy.$

We rewrite this equation in a standard form

$$3x^2(1+\ln y)dx + \left(\frac{x^3}{y} - 2y\right)dy = 0. \quad (*)$$

$M_y = \frac{3x^2}{y}$ $\quad N_x = \frac{3x^2}{y}$, $M_y = N_x \Rightarrow (*)$ is exact.

Therefore, $u(x,y) = \int 3x^2(1+\ln y)dx + T(y) = x^3(1+\ln y) + T(y).$

$\frac{x^3}{y} - 2y = u_y = \frac{x^3}{y} + T'(y) \Rightarrow T'(y) = -2y$, $T(y) = -y^2 + c_1$,

c_1 - constant.

Therefore, $u(x,y) = x^3(1+\ln y) - y^2$ and

$\boxed{x^3(1+\ln y) - y^2 = C}$ is the general solution.

~~(5) $y^2 dx + (e^x - y) dy = 0$.~~

~~$M_y = 2y$ $N_x = e^x$ $M_y \neq N_x$. Therefore we are trying to find an integrating factor.~~

~~(5) $(x^2 + 3\ln y) y\, dx = x\, dy$~~

~~We first rewrite this equation in a standard form:~~

~~$(x^2 + 3\ln y) y\, dx - x\, dy = 0$.~~

~~$M_y = x^2 + \frac{3}{y} \cdot y + 3\ln y = x^2 + 3 + 3\ln y$.~~

~~$N_x = -1$ $M_y \neq N_x$ Therefore we will try to find an integrating factor.~~

(5) $(x^2 + y^2 + x)\, dx + y\, dy = 0$.

$M_y = 2y$ $N_x = 0$ $M_y \neq N_x$. Therefore we will try to find an integrating factor μ.

$\frac{M_y - N_x}{N} = \frac{2y - 0}{y} = 2$ can be viewed as a function of x only (in fact, it is a constant function).

Therefore $\mu = \mu(x) = e^{\int 2dx} = e^{2x + c_1}$, c_1 - constant.

We may take $C_1 = 0$. Thus, an integrating factor is $\mu(x,y) = e^{2x}$. Now we get an exact equation

$$\underbrace{e^{2x}(x^2+y^2+x)}_{\tilde{M}}\,dx + \underbrace{e^{2x}y}_{\tilde{N}}\,dy = 0.$$

$$\left[\text{indeed, } \tilde{M}_y = 2ye^{2x} = \tilde{N}_x\right]$$

Therefore, we look for a function $u = u(x,y)$ in the form $u(x,y) = \int e^{2x} y\,dy + T(x) = \frac{1}{2}e^{2x}y^2 + T(x)$.

$$e^{2x}(x^2+y^2+x) = u_x = e^{2x}\cdot y^2 + T'(x).$$

Therefore, $T'(x) = e^{2x}(x^2+x)$.

Hence, $T(x) = \int e^{2x}(x^2+x)\,dx = \int e^{2x}\cdot x^2 dx + \int e^{2x}\cdot x dx =$

$= \cancel{\int \frac{1}{2}e^{2x}\cdot 2x\,dx}\ \frac{1}{2}e^{2x}x^2 - \int \frac{1}{2}e^{2x}\cdot 2x\,dx + \int e^{2x}\cdot x\,dx =$

$= \frac{1}{2}e^{2x}\cdot x^2 - \int e^{2x}\cdot \cancel{2}x\,dx + \int e^{2x}\cdot x\,dx = \frac{1}{2}e^{2x}\cdot x^2 + C_1$, C_1 – constant.

Therefore, $u(x,y) = \frac{1}{2}e^{2x}\cdot y^2 + \frac{1}{2}e^{2x}\cdot x^2 = \frac{1}{2}e^{2x}(y^2 + x^2)$

Hence the general solution is $\frac{1}{2}e^{2x}(y^2+x^2) = C_2$ or

$$\boxed{e^{2x}(y^2 + x^2) = C}$$

(6) $xy^2(xy'+y)=1$.

First let us rewrite the equation in a standard form: $xy^2\left(x\frac{dy}{dx}+y\right)=1$. Multiplying by dx we get $x^2y^2dy+xy^3dx=dx$. Finally,

$$(xy^3-1)\,dx+x^2y^2\,dy=0. \quad (*)$$

$M_y=3xy^2 \quad N_x=2xy^2$, so $M_y\neq N_x$. Let us find ~~ah~~ an integrating factor $\mu(x,y)$. Since $\frac{M_y-N_x}{N}=$

$$=\frac{3xy^2-2xy^2}{x^2y^2}=\frac{xy^2}{x^2y^2}=\frac{1}{x}$$ is a function of x only

we get that $\mu=\mu(x)=e^{\int\frac{dx}{x}}=e^{\ln x+c}=x\cdot e^c$.

Putting $c=0$ we get $\mu(x)=x$.

Multiplying $(*)$ by $\mu(x)$ we get an exact equation $\underbrace{(x^2y^3-x)}_{\tilde{M}}dx+\underbrace{x^3y^2}_{\tilde{N}}dy=0$.

[indeed, $\tilde{M}_y=3x^2y^2=\tilde{N}_x$]

Now, we are looking for $u(x,y)=\int x^3y^2\,dy+T(x)=$
$=\frac{1}{3}x^3y^3+T(x)$. We know that $x^2y^3-x=u_x=x^2y^3+T'(x)$

Therefore $T'(x)=-x \Rightarrow T(x)=-\frac{1}{2}x^2+C_1$, C_1 - constant.

Thus, $u(x,y)=\frac{1}{3}x^3y^3-\frac{1}{2}x^2$ and $\boxed{\frac{1}{3}x^3y^3-\frac{1}{2}x^2=C \text{ is a general solution.}}$

(7) $\cancel{(2e^{y} - x)\, y' = 1}$

(7) $x^2 y' + xy + 1 = 0.$

Dividing by x^2 we get a linear equation in the standard form:

$$y' + \frac{1}{x} y + \frac{1}{x^2} = 0, \text{ or } y' + \frac{1}{x} y = -\frac{1}{x^2}.$$

$$f(x) = \frac{1}{x}, \quad r(x) = -\frac{1}{x^2}.$$

We know that the general solution is

$$y = e^{-\int f(x)dx} \cdot \left(\int e^{\int f(x)dx} \cdot r(x)\, dx + C \right).$$

(1) $e^{-\int f(x)dx} = e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}.$

(2) $\int e^{\int f(x)dx} \cdot r(x)\, dx = \int e^{\ln x} \cdot \left(-\frac{1}{x^2}\right) dx = -\int \frac{dx}{x} = -\ln|x|.$

Thus, the general solution is

$$\boxed{y = \frac{1}{x}\left(-\ln|x| + C\right)}$$

(8) $y' + 2y = y^2 e^x$ $(*)$.

Let us make a substitution $u(x) = y(x)^{-1}$. Then $y(x) = u(x)^{-1}$. Therefore, $y' = -\frac{u'(x)}{u^2(x)}$. Thus, we rewrite $(*)$ as $-\frac{u'(x)}{u^2(x)} + 2u(x)^{-1} = u(x)^{-2} e^x$.

Multiplying both sides by $-u^2(x)$ we get

$$u'(x) - 2u(x) = -e^x$$ – first-order linear diff. equation in a standard form. Hence

$$u(x) = e^{-\int -2dx} \cdot \left(\int e^{\int -2dx} \cdot (-e^x) dx + C \right) =$$

$$= e^{2x} \cdot \left(-\int e^{-2x} \cdot e^x dx + C \right) = e^{2x} \cdot \left(-\int e^{-x} dx + C \right) =$$

$$= e^{2x} \left(e^{-x} + C \right) = e^x + e^{2x} C.$$

Finally, substituting back $u(x) = y(x)^{-1}$ we get

$$\boxed{y(x) = \frac{1}{e^x + e^{2x} C}}$$ is a general solution.