Page1.pdf

Page2.pdf

Page3.pdf

Page4.pdf

Page5.pdf

Page6.pdf

Page7.pdf

Page1.pdf

Page2.pdf

Page3.pdf

Page1.pdf

page2.pdf

page3.pdf

Page1.pdf

Page2.pdf

Page1.pdf

Page2.pdf

Page3.pdf

FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

Some Definitions

Differential Equation: an equation that contains the derivatives of one or more dependent variables with respect to one or more independent variables.

Ordinary Equation: A differential equation with ordinary (as opposed to partial) derivatives of one or more dependent variables with respect to a *single* independent variable.

Order of an Equation: The highest derivative in an equation.

Linearity: An equation is linear if (a) all the powers of y and its derivatives is 1 (b) There are no nonlinear functions of y present in the equation and (c) The coefficients of y and its derivatives are functions of x only.

Examples: Determine the order and linearity of the following equations. If nonlinear, explain why.

$$(a)\,(1-x)y'''-3x\cos xy''-(3y+4)y'=6x$$

$$(b)\,x^3\frac{d^4y}{dx^4}+x^4\left(\frac{dy}{dx}\right)^3=\ln(xy)$$

$$(c)\,5xy^{(7)}+4x^7y''-\frac{y'}{x^2}=0$$

$$(d)\,x\frac{dy}{dx}+\frac{4}{y}=6$$

Solution:

(a) Order = 3. Linearity: Nonlinear due to coefficient of y'.
(b) Order = 4. Linearity: Nonlinear due to $(y')^3$ and ln(xy) – a nonlinear function of y.
(c) Order = 7. Linearity: Linear.
(d) Order = 1. Linearity: Nonlinear as (4/y) is a nonlinear function of y.

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Solutions to Differential Equations: A function y = f(x) is a solution to a given differential equation if the equation is satisfied if you plug in y and its derivatives.

Example: Show that $y_1 = C_1\cos 3t$ and $y_2 = C_2\sin 3t$ are both solutions to the equation: Is the linear combination of y_1 and y_2 also a solution of y" + 9y = 0?

Solution:

$$y_1 = C_1 \cos 3t$$
$$y_1' = -3C_1 \sin 3t$$
$$y_1'' = -9C_1 \cos 3t$$

$$y_2 = C_2 \sin 3t$$
$$y_2' = 3C_2 \cos 3t$$
$$y_2'' = -9C_2 \sin 3t$$

plug in to check if equation is satisfied :
$$LHS : -9C_1 \cos 3t + 9(C_1 \cos 3t) = 0 = RHS$$

plug in to check if equation is satisfied :
$$LHS : -9C_2 \sin 3t + 9(C_2 \sin 3t) = 0 = RHS$$

Linear Combination :

$$y = C_1 y_1 + C_2 y_2$$
$$y = C_1 \cos 3t + C_2 \sin 3t$$
$$y' = -3C_1 \sin 3t + 3C_2 \cos 3t$$
$$y_1'' = -9C_1 \cos 3t - 9C_2 \sin 3t$$

plug in to check if equation is satisfied :
$$LHS : -9C_1 \cos 3t - 9C_2 \sin 3t + 9(C_1 \cos 3t + C_2 \sin 3t) = 0 = RHS$$

SOLVING FIRST ORDER DIFFERENTIAL EQUATIONS

Separable Equations

Separable equations are often the easiest ones to solve. Given a differential equation you should always first check it is separable before applying other techniques. Solving separable equations is a simple 3 step process:

Step 1: Look to separate the variables: Get the x's on one side and the y's on the other do that you end up with:

$$f(y)dy = f(x)dx$$

Note – a) you can't have dx or dy in the denominator.
b) an equation of the form $dy/dx + p(x)y = f(x)$ must have $p(x)$ and $f(x)$ = constant for it to be separable

Step 2: Integrate both sides of the equation:

$$\int f(y)dy = \int f(x)dx$$

Step 3: Use initial conditions to solve for the constant of integration.

First order differential equations: Integrating Factor

As we know 1st order linear differential equations take the form $C_1(x)\, y' + C_2(x)y = g(x)$. Like separable equations, solving first order equations is a mechanical process and can be solved by following the following 5 steps.

Step 1: Ensure that the coefficient of y' is 1. That is, divide the entire equation by $C_1(x)$ to obtain the following form:

$$y' + p(x)y = f(x)$$

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Step 2: Compute the integrating factor:

$$\mu = e^{\int p(x)dx}$$

Step 3: Multiply the entire equation by μ. By doing so the left hand side of the equation will be the derivative of μy (product rule backwards):

$$\frac{d}{dx}(\mu y) = \mu f(x)$$

Step 4: Integrate both sides and solve for y:

$$\mu y = \int \mu f(x)dx$$
$$y = \frac{1}{\mu}\int \mu f(x)dx$$

Step 5: Use initial conditions to solve for the constant of integration.

Notes – sometimes computing the integral is hard: leave your answer in terms of the integral.
- you do not need to include a constant of integration when computing μ.
- all terms in the final answer that tend to 0 as the independent variable tends to ∞ (i.e. terms with negative exponents) are known as *transient terms*. The rest are called *steady state* terms

Exact Equations

The exact equation is a very particular type of equation and usually very easily recognized by its form:

$$M(x, y)dx + N(x, y)dy = 0$$

There are 2 methods for solving this equation The first method involves the following steps:

Step 1: Make sure that the equation is in the form above and ensure that it is in fact exact, that is:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Step 2:

$$Let \frac{\partial f}{\partial x} = M \qquad Let \frac{\partial f}{\partial y} = N$$

$$So\ f_1 = \int Mdx + g(y) \qquad So\ f_2 = \int Ndy + h(x)$$

Step 3: Now the answer is simply a the sum: $f_1 + f_2 = C$. The only trick here is to be sure to include common terms between f_1 and f_2 only once and you are set.

Method 2: Sometimes one of the two integrations are difficult, in this case method 2 is more convenient to use. The first step of both methods is the same, things differ from the 2nd step onward:

Step 2:

$$Let \frac{\partial f}{\partial x} = M$$

$$So\ f = \int Mdx + g(y)$$

Step 3: Use the answer above to solve the following equation. This will allow you to solve for g(y)

$$Let \frac{\partial f}{\partial y} = N$$

Step 4: Replace g(y) in step 3 and write your solution in the form f(x, y) = C

Optional: Non Exact Equations – Making them Exact.

Sometimes non exact equations can be made exact by first multiplying the entire equation by an integrating factor. The following formulas allow you to calculate the integrating factor, as long as it is calculated to be a function of a single variable, the equation can be made exact.

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}} \qquad \mu(y) = e^{\int \frac{N_x - M_y}{M}}$$

In the equations above M_y and N_x are used to denote the partial derivatives with respect to the subscript.

SOLUTIONS BY SUBSTITUTION:

Many differential equations that initially appear difficult can be simplified considerable by making a simple substitution. There are many such substitutions but three are most common.

TYPE 1: Ax + By + C

Step 1: Let u = Ax + By + C, then u' = A +By', solve for y' and substitute back into the equation. It should simplify into something you know how to solve.

Step 2: Solve the equation and substitute back for u.

TYPE 2: HOMOGENEOUS

let $y = ux$
$dy = u\,dx + x\,du$.

Homogeneous equations usually take the form of exact equations, so after checking whether or not they are exact we can see if they are homogeneous and follow the steps outlined below.

$$M(x, y)dx + N(x, y)dy = 0$$

Step 1: Determine whether or not the equation is homogeneous by substituting all x's with λx and all y's with λy. Factor out the highest power of λ in both M and N and as long as you are able to re-attain your original equation, the equation is homogeneous.

Step 2: Substitute y = ux (or x = uy) and dy = udx + xdu (or dx = udy +ydu) and simplify. Usually if N(x, y) seems harder, make the substitution in brackets. The equation should simplify into a separable equation, which you can now solve.

Step 3: Substitute back and simplify your answer.

TYPE 3: BERNOULLI

let $u = y^{1-n}$.

The Bernoulli equation is the simplest to recognize. The equation takes a form similar to the first order linear differential equation except it has a nonlinear term on the right hand side:

$$\frac{dy}{dx} + p(x)y = f(x)y^{n}$$

Bernoulli equations are generally long to solve so you should practice many of these. They always simplify into first order linear differential equations. The steps to solve these equations are as follows:

Step 1: Make sure the equation is in the correct form (as above)

Step 2: Let $u = y^{1-n}$, solve for y and y' (remember to use the chain rule – you must have a u' term when you solve for y')

Step 3: Substitute and simplify. The equation should reduce to a first order linear differential equation. Solve.

Step 4: Substitute back and simplify.

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Variation of Parameters

When the right hand side is hard, that is it can't be solved using the method of undetermined coefficients, we can use the method of variation of parameters. The method uses the solutions of the homogeneous solution along with f(x) to determine a linearly independent particular solution of the problem. As with much of this course, the process is very methodical:

Step 1: Determine the corresponding homogeneous solution to the given problem. This will give you y_1 & y_2.

Step 2: Using the information from step 1 compute the following determinants – the coefficient of y'' must equal 1.

eq'n must be in the form. $y'' + p(x)y' + q(x)y = f(x)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 3: Find u_1 & u_2 using the following formula.

$$u_1' = \frac{W_1}{W} \rightarrow u_1 = \int \frac{W_1}{W} dx$$

$$u_2' = \frac{W_2}{W} \rightarrow u_2 = \int \frac{W_2}{W} dx$$

Step 4: The particular solution is now calculated as $y_p = u_1y_1 + u_2y_2$. The general solution is the sum of the homogeneous and particular solutions and like always constants are calculated using initial conditions if any.

Cauchy Euler Equations

— when eq'n looks similar to: $3x^3y''' + 2x^2y'' + 6xy + 5y = 0$.
exponents of x = degree of differentiation of y.

Cauchy-Euler equations are very easy both to recognize and to solve. They take the form:

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1 x \frac{dy}{dx} + a_0 y = f(x) \qquad \boxed{y = x^m}$$

That is each derivative has a corresponding power of x as a coefficient. The solution to such an equation is similar to the constant coefficient problem. The auxiliary equation, however, is harder to find by inspection. The steps to solve the problem are outlined below.

Step 1: Make sure the equation is in the form above, if not put it in the form (usually by multiplying by x if f(x) = 0)

Step 2: let $y = x^m$, find the necessary derivatives of y and plug them back into the original equation. Simplify the equation using $x^a x^b = x^{a+b}$. At this point you should be able to factor out an x^m term and find the auxiliary equation.

Step 3: For the second degree case, the following table shows the form of the solution based on the roots of the auxiliary equation. These can be generalized for higher order equations as well.

1. *Distinct real roots*, m_1 & m_2	$: y = C_1X^{m_1} + C_2X^{m_2}$
2. *Repeated real roots*, m_1	$: y = C_1X^{m_1} + C_2X^{m_1}\ln X$
3. *Complex roots*, $a \pm bi$	$: y = X^a(C_1\cos(b\ln X) + C_2\sin(b\ln X))$

Step 4: If the equation is not homogeneous, find the particular solution using variation of parameters. Be sure to transform the equation into the correct form before applying variation of parameters – the coefficient of the highest power of y must be 1 (standard form).

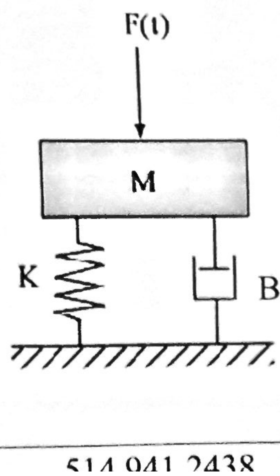
Step 5: Find constants using initial conditions if given.

LINEAR MODELS: 2nd Order

In this section we consider 2 main types of dynamical systems that can be modelled by second order differential equations. The first is the mass-spring-damper system which is a basic building block for many systems, in fact entire courses (MECH 370, MECH 373, MECH 448) deal with this topic. The second is the resistor-inductor-capacitor (RLC) circuit which is dealt with in detail in circuit analysis courses (ELEC 273, ELEC 275).

Mass Spring Damper Systems

The mass spring damper system can be visualized as the following diagram.



Where k is the spring constant of the spring in N/m, m is the mass of the object and B is the damping ratio in Ns/m. Finally, F(t) is the force driving the motion. At the end, solving MSD problems amounts to solving the following second order differential equation:

damping coefficient → ALWAYS GIVEN
spring constant.

$$mx''+Bx'+kx = F(t)$$

some force that drives motion → either given or zero.

MSD systems are classified as follows: free undamped motion ($B=0$, $f(t) = 0$), free damped motion ($f(t) = 0$), and driven motion. The system response is classified according to the discriminant of the auxiliary equation:

distinct →	Discriminant > 0: Over Damped
repeated →	Discriminant = 0: Critically damped
complex →	Discriminant < 0: Under Damped

Template:
Find: $m =$ $F(t) =$
$B =$ $x(0) =$
$K =$ $x'(0)$

signs are COUNTERINTUITIVE.

And Plug into:
$mx'' + Bx' + Kx = F(t)$

Solving MSD problems are usually straight forward. You may have to use Hooke's law (F=kx) to solve for k. If units are given in lbs and feet, then divide the weight by 32 to get m (in slugs). As for initial conditions, anything going up is negative and anything going down is positive, anything starting up is negative and starting down (below equilibrium) is positive.

As the motion is usually oscillatory the solution will often be: $x(t) = c_1\cos(wt) + c_2\sin(wt)$. If the question asks about time, it is often convenient to convert to $x(t) = A\sin(wt + \phi)$ where:

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \tan\phi = c_1/c_2$$

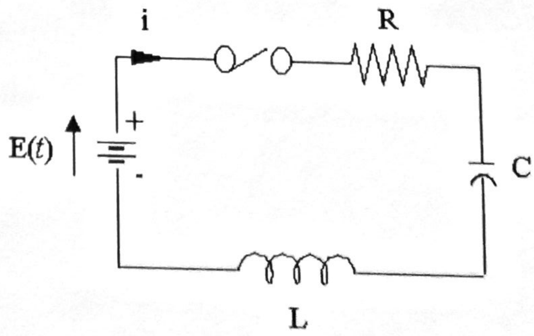
$x(t) = c_1\cos(wt) + c_2\sin(wt)$
⇓
$x(t) = A\sin(wt + \varphi)$.

Let take a look at a couple of examples.

Second Order Circuits: RLC Circuits

This course only deals with the series connection of the RLC circuit. One can solve for the charge q, in this circuit using the following second order equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$



HIGHER ORDER DIFFERENTIAL EQUATIONS

Linear Independence

As we saw in the previous tutorial a given differential equation can have more than one solution and as long as the solutions are linearly independent, the linear combination of the solutions is also a solution. How do we determine whether a set of functions is linearly independent or not? The answer is that we compute the *Wronskian*:

$$W(f_1,\ldots,f_n)=\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{n-1} & f_2^{n-1} & \cdots & f_n^{n-1} \end{vmatrix}$$

as long as the Wronskian $\neq 0$, the functions are linearly independent and form a fundamental set.

Example: determine whether or not the given set of functions is linearly independent.

a) $f_1(x) = 5$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$

Reduction of Order

Any second order differential equation has two linearly independent solutions. If we know one solution before hand, we can compute the second using reduction of order, which basically amounts to calculating y_2 using the following formula:

$$y_2 = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2(x)}dx$$

Notes – As y_1 is a function of x, you can not cancel out the y_1 outside the integral with that inside.

- The general solution of the equation is written as $y = C_1y_1 + C_2y_2$
- The equation must be in the correct form: $y'' + p(x)y' + q(x)y = 0$

means RHS = 0.

Homogeneous Constant Coefficient Linear Equations

Homogeneous constant coefficient linear equations can be solved by finding the solution to the corresponding auxiliary equation, which theoretically is obtained by substituting $y=e^{mx}$ into the given equation, but can easily be found by inspection.

For a second order differential equation, the corresponding auxiliary equation is a quadratic with 3 possible solutions. Each case along with the general solution is presented below

Solve quadratic and use →

1. *Distinct real roots*, m_1 & m_2	: $y = C_1e^{m_1t} + C_2e^{m_2t}$
2. Re*peated real roots*, m_1	: $y = C_1e^{m_1t} + C_2te^{m_1t}$
3. *Complex roots*, $a \pm bi$	: $y = e^{at}(C_1 \cos bt + C_2 \sin bt)$

MEMORIZE!

The equations above should be memorized. The steps to solve these equations are simple; first find the auxiliary equation by inspection, find its solution and write out the general solution to the equation using the equations above. Finally use any initial conditions given to solve for the constants C_1 and C_2.

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Solving by Method of Undetermined Coefficients

What happens when the equation we are trying to solve is not homogeneous (ie the right hand side is non zero)? In this case the general solution is given by the sum of the corresponding homogeneous solution, y_h, and the particular or complementary solution, y_p, based on the right hand side. The method of undetermined coefficients is the easy way to go as long as the right hand side is a simple function. The following table (which you must memorize) shows you how to choose y_p based on f(x) – the right hand side.

f(x)	y_p
3	A
8x-9	Ax + B
$4x^2-3x$	Ax^2+Bx+C
$7x^3-4x+6$	Ax^3+Bx^2+Cx+D
sin3x or cos3x	Asin3x + Bcos3x
e^{-7x}	Ae^{-7x}
$(5x+4)e^{3x}$	$(Ax+B)e^{3x}$
x^2e^{2x}	$(Ax^2+Bx+C)e^{2x}$
$x^3\sin6x$ or $x^3\cos6x$	$(Ax^3+Bx^2+Cx+D)\sin6x+(Ex^3+Fx^2+Gx+H)\cos6x$
$e^{6x}\cos5x$	$Ae^{6x}\cos5x + Be^{6x}\sin5x$
$x^2e^{4x}\cos6x$	$(Ax^2+Bx+C)e^{4x}\cos6x+(Dx^2+Ex+F)e^{4x}\sin6x$

The solution of a non-homogeneous constant coefficient differential equation using the method of undetermined coefficients can be found using the following steps.

Step 1: Solve the corresponding homogeneous differential equation: y_h

Step 2: Choose the form for y_p by comparing f(x) with the table above. Make sure that no term in the particular solution already shows up in the homogeneous solution. If it does then eliminate the redundancy by multiplying y_p by x in order to form an linearly independent set.

Step 3: Compute the necessary derivatives of y_p and substitute these back into the given differential equation. And simplify by grouping the coefficients of like terms.

Step 4: Solve for the undetermined coefficients by comparing the right and left hand sides of the equation.

Step 5: Write the final solution in the form $y = y_h + y_p$. Use any given initial conditions to find constants C_1 and C_2.

SERIES SOLUTIONS OF DIFFERENTIAL EQUATIONS

So far we have mainly looked at linear constant coefficient differential equations. Only the Cauchy Euler equation had variable coefficients. However, equations with variable coefficients even as simple as y" + xy = 0 are considerably harder to solve. The main technique used to solve them involves using power series.

Adding power series

Before we begin anything else, we must first learn to add power series. This is a three step process as demonstrated in the following example.

Example: Group the following power series

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1}$$

Step 1: Remove necessary terms so that the power on x is the same in all series once the first number is plugged in.

In our case if we plug in n=2 in the first term we get x^0, where as if we plug in n=0 in the second term we get x^1. Clearly the powers of x are not the same, however if we remove the first term from the first power series, then both terms will start at x^1 (you always want to start at the highest power of x):

$$2(2-1)C_2 x^{2-2} + \sum_{n=3}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1}$$

$$2C_2 x^0 + \sum_{n=3}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1}$$

Note that the first series starts at n=3

Step 2: Make a change in the index of summation so that the summation starts at 1. That is let k=the power of x. Do this separately for each series.

Thus, we let k=n-2 for the first series and k=n+1 for the second.

Making the substitutions we get:

$$2C_2 + \sum_{k=1}^{\infty}(k+2)(k+2-1)C_{k+2}x^k + \sum_{k=1}^{\infty}C_{k-1}x^k$$

Step 3: Group the two series together, factor out an x^k term and come up with an equation relating the coefficients.

$$2C_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)C_{k+2}x^k + C_{k-1}x^k]$$

$$2C_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)C_{k+2} + C_{k-1}]x^k$$

$$Recurssion\ Relationship: (k+2)(k+1)C_{k+2} + C_{k-1}$$

Power Series Solutions

Solving differential equations by means of a power series solution involves substituting

$$y = \sum_{n=0}^{\infty}C_n x^n \quad y' = \sum_{n=1}^{\infty}nC_n x^{n-1} \quad y'' = \sum_{n=2}^{\infty}n(n-1)C_n x^{n-2}$$

into the given differential equation and adding the power series as shown above. At this point we can come up with a recurrence relationship between the coefficients by making the equation we get at step 3 equal to zero.

At this point we substitute various values for k and try and find a pattern for the coefficients. We can usually express subsequent coefficients in terms of C_0 and C_1. Grouping these in turn will give 2 power series solutions to the equation. Let's see how this is done via the following examples.

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Matrix Solutions

We are familiar of solving a system of simple equations using matrices in a linear algebra course. The idea is to convert the system into matrix form and reduce the matrix using Gauss Jordan elimination until a solution is achieved. Although the idea can be generalized to solving a system of differential equations, the method is not exactly the same. Instead of row reducing, we find the eiganvalues and corresponding eiganvectors of the system in order to achieve a solution. The steps are outlined as follows:

Step 1: Write the system in matrix form: $\mathbf{X}' = \mathbf{A}\mathbf{X}$

Step 2: Compute the eiganvalues of matrix **A** by finding the roots of the characteristic equation:

$$\mathrm{Det}(A-\lambda I) = 0$$

Step 3: Find the corresponding eigenvector, **K** to each eiganvalue, [0 0] is never an acceptable eiganvector. This is done by plugging in λ and finding the solution to $A-\lambda I=0$.

Step 4: Based on the nature of the eiganvalues (distinct, repeated, and complex) we can follow more steps to reach the solution. For the two dimensional case:

Distinct Eiganvalues: $\mathbf{X} = C_1\mathbf{K}_1e^{(\lambda 1)t} + C_2\mathbf{K}_2e^{(\lambda 2)t}$

Repeated Eiganvalues: Here it depends whether there are two eiganvectors or only one corresponding to λ.

If there are 2 eiganvectors then the solution is:

$$\mathbf{X} = C_1\mathbf{K}_1e^{\lambda t} + C_2\mathbf{K}_2e^{\lambda t}$$

If however, only one eigenvector exists then a second vector, **P**, can be found by solving

$$(\mathbf{A}-\lambda\mathbf{I})\mathbf{P} = \mathbf{K}$$

And the solution will take the form:

$$\mathbf{X} = C_1\mathbf{K}_1e^{\lambda t} + C_2[\mathbf{K}_1te^{\lambda t} + \mathbf{P}e^{\lambda t}]$$

Complex Eiganvalues $a\pm ib$: The solution takes on the following form:

$$X = C_1[B_1\cos(bt)-B_2\sin(bt)]e^{at} + C_2[B_2\cos(bt)+B_1\sin(bt)]e^{at}$$

Where B_1 and B_2 are the real and imaginary coefficients of the eigenvector $K = B_1+iB_2$

Lets take a look at an example of each type:

Examples:

(a) $x' = (-5/2)x + 2y, \quad y' = (3/4)x - 2y$
(b) $x' = -6x + 5y, \quad y' = -5x + 4y$
(c) $x' = 6x - y, \quad y' = 5x + 2y$

Non Homogeneous Linear Systems

Undetermined Coefficients

The method of undetermined coefficients when solving a system of equations is very similar to the method we are already used to. The steps are straight forward: calculate the solution to the corresponding homogeneous system and then choose a form for the particular solution, plug it in and solve for the undetermined coefficients. The best thing to is look at examples to understand how this works exactly.

Examples

(a) $x' = 6x + y + 6t$, $y' = 4x + 3y - 10t + 4$, Given that $\mathbf{X_h} = C_1(1\ -4)^T e^{2t} + C_2(1\ 1)^T e^{7t}$
(b)

Variation of Parameters

Step 1: Solve the corresponding homogeneous equation and get $\mathbf{X_1}$ and $\mathbf{X_2}$

Step 2: Compute the inverse of $\mathbf{V(t)=(X_1\ X_2)}$

Step 3: The particular solution can now be obtained using the following formula:

$$X_p = V(t)\int V^{-1}(t)F(t)dt$$

Examples: Solve the given system of equations.

$$(a)\quad X' = \begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix} X + \begin{pmatrix} 3t \\ e^{-t} \end{pmatrix}$$

$$(b)\quad X' = \begin{pmatrix} 1 & 8 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} e^{-t} \\ te^{t} \end{pmatrix}$$