Final Examination

Math1339 (A) Calculus and Vectors

December 12, 2009

14:00-17:00

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Final Examination

MAT 1339 A

Instructor: Jie Sun

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Name: ______________________________

Student Number: ____________________

- There are eight questions worth 100 points in total.
- Only non-programmable, non-graphic calculators are permitted.
- Please answer the questions in the provided space, clearly stating which question you are answering. Use the back of the pages if necessary, but please indicate you are doing so.

Good Luck!

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total
/10	/15	/15	/10	/15	/10	/15	/10	/100

Problem 1: (10 points)

Let $f(x) = \begin{cases} \frac{x^2-9}{x+3}, & \text{for } x \neq -3 \\ c, & \text{for } x = -3. \end{cases}$

For what value of c is $f(x)$ continuous at $x = -3$? Justify your answer.

Solution (10 points): $f(x)$ is continuous at $x = -3$ if and only if $\lim_{x \to (-3)} f(x) = f(-3) = c$.

$$\lim_{x \to (-3)} f(x) = \lim_{x \to (-3)} \frac{x^2 - 9}{x + 3} = \lim_{x \to (-3)} \frac{(x-3)(x+3)}{x+3} = \lim_{x \to (-3)} x - 3 = -6.$$

Therefore, $c = -6$.

Problem 2: (15 points)

Find the derivative of the following functions. Do not simplify.

(a) $y = (4x^3 + 2x + 1)e^{\sin 2x}$.

(b) $y = \frac{\cos 3x}{2^x + 6^x}$.

Solution (a) (7 points):

$$\begin{aligned} y' &= (4x^3 + 2x + 1)' e^{\sin 2x} + (4x^3 + 2x + 1)(e^{\sin 2x})' \\ &= (12x^2 + 2)e^{\sin 2x} + (4x^3 + 2x + 1)(e^{\sin 2x} \cos 2x \cdot 2). \end{aligned}$$

Solution (b) (8 points):

$$\begin{aligned} y' &= \frac{(\cos 3x)'(2^x + 6^x) - (\cos 3x)(2^x + 6^x)'}{(2^x + 6^x)^2} \\ &= \frac{(-\sin 3x \cdot 3)(2^x + 6^x) - (\cos 3x)((\ln 2)2^x + (\ln 6)6^x)}{(2^x + 6^x)^2}. \end{aligned}$$

Problem 3: (15 points)

Let $f(x) = \frac{x^2-x+6}{x^2-x-12}$.

(a) Determine the domain of the function.

(b) Find the vertical asymptotes.

(c) Determine all local extreme points, if any exists.

(d) Find the intervals where $f(x)$ is concave up, concave down.

(e) Sketch the graph of the function.

Solution (a) (3 points): The denominator must not be zero: $x^2 - x - 12 = (x-4)(x+3) \neq 0$, thus $x \neq 4$ and $x \neq -3$. So the domain of the function is $\{x \in \mathbb{R} \mid x \neq 4, x \neq -3\}$.

Solution (b) (3 points): We have $\lim_{x\to 4^+} f(x) = +\infty$ and $\lim_{x\to 4^-} f(x) = -\infty$, so $x = 4$ is a vertical asymptote. Similarly $\lim_{x\to(-3)^+} f(x) = -\infty$ and $\lim_{x\to(-3)^-} f(x) = +\infty$ implies that $x = -3$ is a vertical asymptote.

Solution (c) (3 points): Let

$$f'(x) = \frac{(2x-1)(x^2-x-12)-(x^2-x+6)(2x-1)}{(x^2-x-12)^2} = \frac{-36x+18}{(x^2-x-12)^2} = 0,$$

we get $x = \frac{1}{2}$. When $x > \frac{1}{2}$, $f'(x) < 0$. When $x < \frac{1}{2}$, $f'(x) > 0$. Thus $(\frac{1}{2}, \frac{-23}{49})$ is a local maximal point.

Solution (d) (3 points): We can calculate

$$f''(x) = \frac{-36(x^2-x-12)^2-(-36x+18)2(x^2-x-12)(2x-1)}{(x^2-x-12)^4} = \frac{36(3x^2-3x+13)}{(x^2-x-12)^3}.$$

Notice that $36(3x^2 - 3x + 13) > 0$ for all x in the domain. Thus when $x > 4$ or $x < -3$, $f''(x) > 0$ and $f(x)$ is concave up. When $-3 < x < 4$, $f''(x) < 0$ and $f(x)$ is concave down.

Solution (e) (3 points):

Problem 4: (10 points)

A farmer has 120 meters of fencing with which he plans to make a rectangular cow pen. The pen is to be divided into 2 sections by a fence running parallel to one of the sides. Find the dimensions that produce the pen of maximum area if the length of the largest section is to be twice the length of the smaller section.

Solution (10 points): Let A be the area of the pen and let P be the perimeter of the pen. Let $2l$ be the length of the largest section, then l is the length of the smallest section. Let w be the width of the pen. Thus

$$A = 3lw \text{ and } P = 6l + 3w = 120.$$

From $6l + 3w = 120$ we get $w = 40 - 2l$. Then

$$A(l) = 3l(40 - 2l) = 120l - 6l^2.$$

The domain of $A(l)$ is $0 < l < 20$. $A'(l) = 120 - 12l$. Let $A'(l) = 0$ we get the critical number of $A'(l)$ is $l = 10$. We can check that $A'(l) > 0$ when $0 < l < 10$ and $A'(l) < 0$ when $10 < l < 20$. Thus $A(l)$ is increasing when $0 < l < 10$ and $A(l)$ is deceasing when $10 < l < 20$. So $A(l)$ has maximum value when $l = 10$ and $w = 40 - 2l = 20$. Therefore, to produce the maximum area of the pen, the length of the pen should be $3l = 30$ meters and the width of the pen should be $w = 20$ meters.

Problem 5: (15 points)

Let $\vec{u} = [3, 2, -1]$, $\vec{v} = [4, -6, 3]$ and $\vec{w} = [1, -2, 2]$. Let θ be the angle between $\vec{u}$ and $\vec{v}$.

(a) Find the dot product $\vec{u} \cdot \vec{v}$.

(b) Find the cross product $\vec{u} \times \vec{v}$.

(c) Find $\cos\theta$ and $\sin\theta$.

(d) Find the projection $\text{proj}_{\vec{u}}\,\vec{v}$ of the vector $\vec{v}$ on $\vec{u}$.

(e) Find the volume of the parallelepiped defined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Solution (a) (3 points):

$$\vec{u} \cdot \vec{v} = -3.$$

Solution (b) (3 points):

$$\vec{u} \times \vec{v} = [0, -13, -26].$$

Solution (c) (3 points):

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{-3}{\sqrt{14}\sqrt{61}} = \frac{-3}{\sqrt{854}}$$

$$\sin\theta = \frac{|\vec{u} \times \vec{v}|}{|\vec{u}||\vec{v}|} = \frac{\sqrt{(-13)^2 + (-26)^2}}{\sqrt{14}\sqrt{61}} = \frac{\sqrt{845}}{\sqrt{854}}.$$

Solution (d) (3 points):

$$\text{proj}_{\vec{u}}\,\vec{v} = \left(\frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u} = \left(\frac{-3}{14}\right)[3, 2, -1] = \left[\frac{-9}{14}, \frac{-6}{14}, \frac{3}{14}\right].$$

Solution (e) (3 points):

$$|\vec{w} \cdot (\vec{u} \times \vec{v})| = |[1, -2, 2] \cdot [0, -13, -26]| = |-26| = 26.$$

Problem 6: (10 points)

(a) Find the vector equation of the line in two-space which is perpendicular to $4x - 3y = 9$ and passes through the point $(1, -2)$.

(b) Find the parametric equation of the line in three-space which is parallel to the x-axis and passes through the point $(3, 8, -1)$.

Solution (a) (5 points): We know that $\vec{n} = [4, -3]$ is a normal vector of $4x - 3y = 9$. So $\vec{n}$ is perpendicular to $4x - 3y = 9$ and $\vec{n}$ gives a direction vector of the line we want. The vector equation of the line is

$$[x, y] = [1, -2] + t[4, -3],$$

where $t \in \mathbb{R}$ is a scalar.

Solution (b) (5 points): The vector $[1, 0, 0]$ is parallel to the x-axis, thus gives a direction vector of the line we want. The parametric equation of the line is

$$\begin{aligned} x &= 3 + t \\ y &= 8 + 0t \\ z &= -1 + 0t, \end{aligned}$$

where $t \in \mathbb{R}$ is a parameter.

Problem 8: (10 points)

Indicate whether the statement is true or false. Justify your answer.

(a) If $\text{proj}_{\vec{v}}\,\vec{u} = \text{proj}_{\vec{u}}\,\vec{v}$, then $\vec{u} \cdot \vec{v} = 0$.

Solution (a) (2 points): False. If $\vec{u} = \vec{v} \neq \vec{0}$, then $\text{proj}_{\vec{v}}\,\vec{u} = \text{proj}_{\vec{u}}\,\vec{v}$, but $\vec{u} \cdot \vec{v} \neq 0$.

(b) All normal vectors of a line in three-space are parallel to each other.

Solution (b) (2 points): False. Consider the line which is parallel to the z-axis. Both $[1, 0, 0]$ and $[0, 1, 0]$ are normal vectors to this line, but they are not parallel to each other.

(c) $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$.

Solution (c) (2 points): False. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

(d) $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

Solution (d) (2 points): True. $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$, thus $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

(e) In three-space, a system of two planes cannot have exactly one solution.

Solution (e) (2 points): True. In three-space, a system of two planes either has infinitely many solutions (when the two planes intersect in a line or the two planes are coincident) or has no solution (when the two planes are parallel and distinct).