Model Solutions

1. Determine whether or not the following arguments are valid. If they are valid, then state the rules of inference used to prove validity. If they are invalid, outline precisely why they are invalid.

    a. If I have parents who are martians then I must live on Mars. I live on Mars. Therefore I must have parents who are martians.

Let p = "I have martian parents" and m = "I live on Mars". The argument is then:

$p \rightarrow m$
m
$\therefore p$

Consider the case where I live on Mars and my parents are earthlings (i.e., not martians). For this case m would be true (since I live on Mars) and $p \rightarrow m$ would also be true (since the antecedent is false). For this case both premises are true but the conclusion is false, so the argument is invalid.

    b. If I can drive a car or ride a bicycle, then I can commute to work. Anyone that rides a bicycle does not have a parking space. I don't have a parking space. Therefore I must ride a bicycle.

Let D(x) be "x drives a car", B(x) be "x rides a bike", C(x) be "x commutes" and P(x) be "x has a parking spot". The argument is then:

$D(me) \vee B(me) \rightarrow C(me)$
$\forall x\ B(x) \rightarrow \neg P(x)$
$\neg P(me)$
$\therefore B(me)$

Universal instantiation would allow the second line to become $B(me) \rightarrow \neg P(me)$ and then this problem is the same as 1a. Consider the case where I neither drive nor bike and I don't have a parking space. For this case $\neg P(me)$ would be true (since I don't have a parking space), $B(me) \rightarrow \neg P(me)$ would be true (since the antecedent is false), and $D(me) \vee B(me) \rightarrow C(me)$ would be true (since the antecedent is false). For this case all premises are true but the conclusion is false, so the argument is invalid.

c. Everyone who has a PC plays computer games. Everyone student taking COMP1501 next semester plays computer games. Therefore every student taking COMP1501 next semester has a PC.

Let P(x) be "x has a pc", G(x) be "x plays games ", C(x) be "x is taking 1501". The argument is then:

$\forall x\ P(x) \rightarrow G(x)$
$\forall x\ C(x) \rightarrow G(x)$
$\therefore \forall x\ C(x) \rightarrow P(x)$

Universal instantiation would allow the first and second lines to become P(me) → G(me) and C(me) → G(me), respectively. Consider the case where I do not have a PC but I do play computer games and I am taking 1501. For this case P(me) → G(me) would be true (since the antecedent is false) and C(me) → G(me) would also be true because both the antecedent and consequent are true. The conclusion C(me) → P(me), however, would be false because the antecedent is true and the consequent is false. For this case all premises are true but the conclusion is false, so the argument is invalid.

For your own edification, here is a case that I might have tried but that wouldn't have worked. Consider the case where I do not have a PC and I do not play games but I am taking 1501. Although P(me) → G(me) would still be true because of the false antecedent, C(me) → G(me) would be false because the antecedent is true but the consequent is false. Since argument validity only applies when the premises are true, I cannot use this case to show invalidity because not all the premises are true in this case.

d. All fish swim. All animals that swim can breathe water. I am an animal and I can swim. Therefore I must be a fish.

Let F(x) be "x is a fish", S(x) be "x swims ", A(x) be "x is an animal", and B(x) be "x breathes water". The argument is then:

$\forall x\ F(x) \rightarrow S(x)$
$\forall x\ A(x) \land S(x) \rightarrow B(x)$
$A(me) \land S(me)$
$\therefore F(me)$

Universal instantiation would allow the first and second lines to become F(me) → S(me) and A(me) ∧ S(me) → B(me), respectively. Consider the case where I am not a fish but I am an animal and I can swim and breathe water. For this case F(me) → S(me) would be true (since the antecedent is false) and A(me) ∧ S(me) → B(me) would also be true because both the antecedent and consequent are true - I was given the antecedent as the third line which is also true because I am an animal that can swim. The conclusion F(me), however, would be false because I am not a fish. For this case all premises are true but the conclusion is false, so the argument is invalid.

2. The floor of a real number x, denoted $\lfloor x \rfloor$, is the unique integer that satisfies the expression $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$.

   a. Compute $\lfloor 1.25 \rfloor, \lfloor 1.75 \rfloor, \lfloor -0.25 \rfloor, \lfloor \lfloor 0.3 \rfloor + \lfloor 0.3 \rfloor + \lfloor 0.3 \rfloor + 2 \rfloor$

   $\lfloor \mathbf{1.25} \rfloor = \mathbf{1}$

   $\lfloor \mathbf{1.75} \rfloor = \mathbf{1}$

   $\lfloor \mathbf{-0.25} \rfloor = \mathbf{-1}$

   $\lfloor \lfloor \mathbf{0.3} \rfloor + \lfloor \mathbf{0.3} \rfloor + \lfloor \mathbf{0.3} \rfloor + \mathbf{2} \rfloor = \lfloor \mathbf{0 + 0 + 0 + 2} \rfloor = \mathbf{2}$

   b. Prove or disprove that $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor - 1$.

   Proof (disproof) by Counterexample. Let x = 0.1 and let y = 0.1.

$$\lfloor \mathbf{0.1} \rfloor + \lfloor \mathbf{0.1} \rfloor \leq \lfloor \mathbf{0.1 + 0.1} \rfloor - \mathbf{1}$$

$$\mathbf{0 + 0} \leq \lfloor \mathbf{0.2} \rfloor - \mathbf{1}$$

$$\mathbf{0} \leq \mathbf{-1}$$

3. What is the power set of $\{2, 4, 8\}$?

$\mathcal{P}\{2, 4, 8\}$ = { ∅, {2}, {4}, {8}, {2, 4}, {4, 8}, {2, 8}, {2, 4, 8} }

4. Prove that $\sqrt{4}+\sqrt{3}+\sqrt{2}$ is an irrational number.

Proof by Contradiction
i.e., assume that $\sqrt{4}+\sqrt{3}+\sqrt{2}$ is rational and find a contradiction.

$\sqrt{4}+\sqrt{3}+\sqrt{2}$ is rational	
$2+\sqrt{3}+\sqrt{2}$ is rational	by math
$\sqrt{3}+\sqrt{2}$ = rational + -2	by math
$\sqrt{3}+\sqrt{2}$ is rational	by the fact that rationals are closed under addition
$\left(\sqrt{3}+\sqrt{2}\right)^2$ is rational	by the fact that rationals are closed under multiplication
$\frac{\left(\sqrt{3}+\sqrt{2}\right)^2}{2}$ is rational	by the fact that rationals are closed under multiplication
$\frac{\left(\sqrt{3}+\sqrt{2}\right)^2}{2}+-\frac{5}{2}$ is rational	by the fact that rationals are closed under addition
$\frac{\sqrt{3}^2+2\sqrt{6}+\sqrt{2}^2}{2}+-\frac{5}{2}$ is rational	by math
$\frac{2\sqrt{6}+5}{2}+-\frac{5}{2}$ is rational	by math
$\sqrt{6}$ is rational	by math
$\sqrt{6}=\frac{a}{b}$	by definition, where $\frac{a}{b}$ is Ratio of Integers in Simplest Form
$6=\frac{a^2}{b^2}$	by math
$6b^2 = a^2$	by math
$2(3b^2) = a^2$	by math
$2c = a^2$	where c = $3b^2$
a^2 is an Even number	by the definition of even numbers

to get to the next step you need a property that you must prove separately

it is NOT a property that you were given in advance

Prove a^2 is an Even number $\rightarrow$ a is an Even number.

Proof by Contradiction; i.e., Prove a is an Odd number $\rightarrow$ a^2 is an Odd number.

a **is an Odd number**

$a = 2d + 1$

$a^2 = (2d + 1)(2d + 1)$

$a^2 = 4d^2 + 4d + 1$

$a^2 = 2(2d^2 + 2d) + 1$

$a^2 = 2e + 1$

a^2 **is an Odd number**

a^2 **is an Even number $\rightarrow$ a is an Even number is now Proven (by Contradiction)**

now you can resume your original proof

a **is an Even number**	**by the property we just proved**
$a = 2f$	
$6b^2 = (2f)^2$	**by math**
$6b^2 = 4f^2$	**by math**
$3b^2 = 2f^2$	**by math**
$3b^2 = 2g$	**where g = f^2**
b^2 **is an Even number**	**by the definition of Even numbers**

b is an Even number **by the property we just proved**

Contradiction

If $\frac{a}{b}$ is its Simplest Form, then a and b can't both be Even or else you could Simplify $\frac{a}{b}$ further by Dividing both the Numerator and the Denominator by 2 and then $\frac{a}{b}$ obviously wouldn't be in Simplest Form

5. Prove that $8 + \frac{4}{\sqrt{3}}$ is an irrational number.

Proof by Contradiction
i.e., assume that $8 + \frac{4}{\sqrt{3}}$ is rational and find a contradiction.

$8 + \frac{4}{\sqrt{3}}$ is rational

$\frac{4}{\sqrt{3}}$ = rational $+ -8$ **by math**

$\frac{4}{\sqrt{3}}$ is rational **by the fact that rationals are closed under addition**

$\frac{1}{\sqrt{3}}$ = rational * $\frac{1}{4}$ **by math**

$\frac{1}{\sqrt{3}}$ is rational **by the fact that rationals are closed under multiplication**

$\frac{1}{\sqrt{3}} = \frac{a}{b}$ **by definition, where $\frac{a}{b}$ is Ratio of Integers in Simplest Form**

$\sqrt{3} = \frac{b}{a}$ **by math**

$\sqrt{3}$ is rational **by definition**

$\sqrt{3} = \frac{a}{b}$ **by definition, where $\frac{a}{b}$ is Ratio of Integers in Simplest Form**

$3 = \frac{a^2}{b^2}$ **by math**

$3b^2 = a^2$ by math

$3c = a^2$ where c = b^2

a^2 is divisible by 3 by the definition of Divisible

to get to the next step you need a property that you must prove separately

it is NOT a property that you were given in advance

Prove a^2 is Divisible by 3 $\rightarrow$ a is Divisible by 3

Proof by Contradiction; i.e., Prove a is not Divisible by 3 $\rightarrow$ a^2 is not Divisible by 3.

n.b., there are Two Cases for a is not Divisible by 3

$$a = 3c + 1 \vee a = 3c + 2$$

Case 1: $a = 3c + 1$

$a^2 = (3c + 1)(3c + 1)$

$a^2 = 9c^2 + 6c + 1$

$a^2 = 3(3c^2 + 2c) + 1$

$a^2 = 3d + 1$

a^2 is not Divisible by 3

Case 2: $a = 3c + 2$

$a^2 = (3c + 2)(3c + 2)$

$a^2 = 9c^2 + 12c + 4$

$a^2 = 9c^2 + 12c + 3 + 1$

$a^2 = 3(3c^2 + 4c + 1) + 1$

$a^2 = 3d + 1$

a^2 is not Divisible by 3

a^2 is Divisible by 3 $\rightarrow$ a is Divisible by 3 is now Proven (by Contradiction)

now you can resume your original proof

a is divisible by 3	by the property we just proved
$a = 3c$	
$3b^2 = (3c)^2$	by math
$3b^2 = 9c^2$	by math
$b^2 = 3c^2$	by math
$b^2 = 3d$	where $d = c^2$
b^2 is Divisible by 3	by the definition of Divisible
b is divisible by 3	by the property we just proved

Contradiction

If $\frac{a}{b}$ is its Simplest Form, then a and b can't both be Divisible by 3 or else you could Simplify $\frac{a}{b}$ further by Dividing both the Numerator and the Denominator by 3 and then $\frac{a}{b}$ obviously wouldn't be in Simplest Form

6. List explicitly the members of the following sets.

   a. $\{i \mid 0 \leq i < 20 \text{ and either } i \text{ begins with either a vowel or the letter } 's' \}$

   {one, six, seven, eight, eleven, sixteen, seventeen, eighteen}

   b. $\{j \mid j \text{ is a platonic solid with less than } 10 \text{ faces}\}$
   {tetrahedron, hexahedron, octahedron}

   c. $\{j \mid j \text{ is the first name of a } COMP1805 \text{ teaching assistant that starts with a vowel}\}$
   {asad} or {asad, yin}

7. Let $S = \{2, 4, \{2,4\}, \{6\}, \{2, 4, 6\}\}$ and T$= \{\{4, 6, 8, 10, 12, 14, 16, 18\}, \{4\}, 2\}$ and answer the following questions.

   a. Which set has the larger cardinality?
   S has cardinality 5 and T has cardinality 3 so S is larger

   b. What is the intersection $S \cap T$?
   {2}

   c. What is the cardinality of the union $S \cup T$?
   7

8. Determine whether or not the following are valid. Justify your answer by using either set identities or membership tables.

   a. $(A - B) \cup (B - A) = (A \cup B \cup C) - (A \cap B) - (C - A)$

A	B	C	$A - B$	$B - A$	$(A - B) \cup (B - A)$	$(A \cup B \cup C)$	$(A \cap B)$	$(C - A)$	$(A \cup B \cup C) - (A \cap B)$	$(A \cup B \cup C) - (A \cap B) - (C - A)$
1	1	1	0	0	0	1	1	0	0	0
1	1	0	0	0	0	1	1	0	0	0
1	0	1	1	0	1	1	0	0	1	1
1	0	0	1	0	1	1	0	0	1	1
0	1	1	0	1	1	1	0	1	1	0
0	1	0	0	1	1	1	0	0	1	1
0	0	1	0	0	0	1	0	1	1	0
0	0	0	0	0	0	0	0	0	0	0

As a "sanity check" you can look at where these differ; an element of both B and C that is not an element of A will be an element of B that won't be removed with the difference of B-A, so it definitely remains a member of the union that is the left-hand side. However, this element would be an element of C that won't be removed with the difference of C-A, meaning that the right-hand side (that could be written as $X-(C-A)$) definitely would not contain this element.

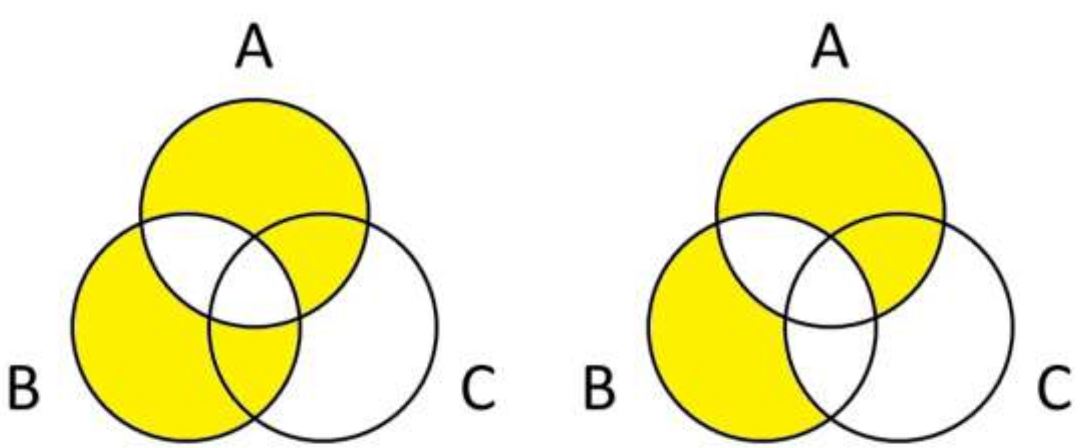
b. $(A \cup B \cup C) - B - (C \cap A) = (A \cap B) \cup (A \cap C)$

A	B	C	$A \cup B \cup C$	$(A \cup B \cup C) - B$	$(C \cap A)$	$(A \cup B \cup C) - B - (C \cap A)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	0	1	0	1	1	1
1	1	0	1	0	0	0	1	0	1
1	0	1	1	1	1	0	0	1	1
1	0	0	1	1	0	1	0	0	0
0	1	1	1	0	0	0	0	0	0
0	1	0	1	0	0	0	0	0	0
0	0	1	1	1	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0

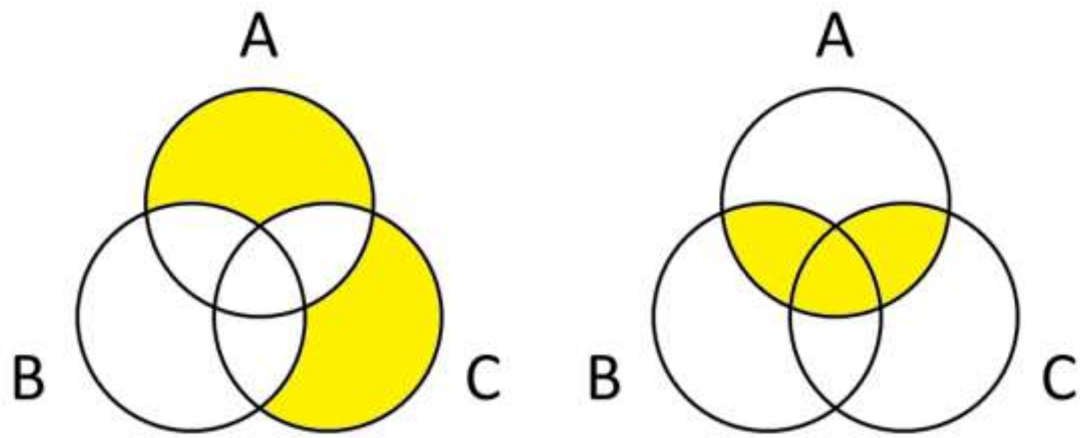
This one is even easier. As a "sanity check" you can look at an element in the intersection of all three. Such an element would definitely be in $(A \cap B) \cup (A \cap C)$, but since it is in the intersection of C and A it would be removed by the second difference operation on the left-hand side.

9. Draw the Venn Diagrams for the following sets.

   a. $(A - B) \cup (B - A) = (A \cup B \cup C) - (A \cap B) - (C - A)$

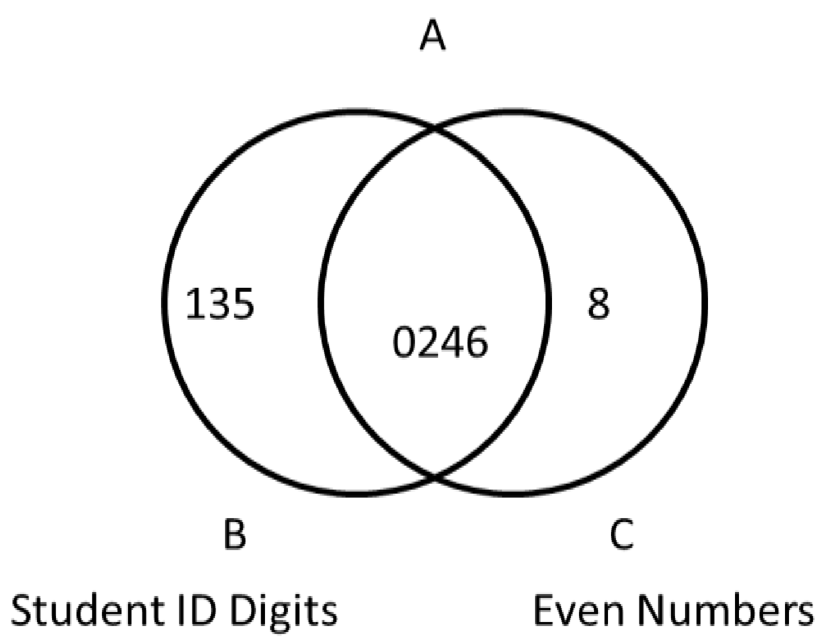


b. $(A \cup B \cup C) - B - (C \cap A) = (A \cap B) \cup (A \cap C)$



10. What is the intersection, A, of the set of all the digits that appear in your student number, B, and the set of all even numbers, C. Draw the Venn Diagram for sets A, B, and C.

Obviously this changes with your student number, but if your student number was 100123456...



11. What is the intersection, X, of the set of all the characters that appear in your last name, Y, and the set of all characters that appear in the set of hexadecimal digits, Z. Draw the Venn Diagram for sets X, Y, and Z.

Obviously this changes with your last name, but if you were lucky enough to have "Collier" as your last name...