Model Solutions

1. Let p be the proposition "I am a student", q be the proposition "I have lots of free time", and r be the proposition "I am tired". Translate the following expressions into English.

   a. $\neg q$

   I do not have lots of free time.

   b. $q \wedge \neg r$

   I have lots of free time and I am not tired.

   c. $p \rightarrow (\neg q \wedge r)$

   If I am a student then I do not have lots of free time and I am tired.

   d. $p \oplus \neg p$

   I am a student or I am not a student but not both.

2. Translate the following English expressions into logical statements. You must explicitly state what the atomic propositions are (e.g., "Let p be the proposition ...") and then show their logical relation.

   a. I read books and I watch movies.

   b = I read books
   m = I watch movies

   $$b \wedge m$$

   b. I am male or I am Canadian.

   m = I am male
   c = I am Canadian

   $$m \vee c$$

c. If it is cloudy outside then it will rain and if it rains then I will get wet.

c = it is cloudy
r = it is raining
w = I get wet

$$(c \rightarrow r) \wedge (r \rightarrow w)$$

d. Either I was born in a month ending in the letter "Y" or I was not

y = I was born in a month ending in y

$$y \oplus \neg y \text{ (alternatively, } y \vee \neg y)$$

3. Determine which of the following statements are True and explain why or why not.

a. 1 < 2 and 2 < 3 and 1 < 3

$$(1 < 2) \wedge (2 < 3) \wedge (1 < 3)$$
$$\text{True} \wedge \text{True} \wedge \text{True}$$
$$\text{True} \wedge \text{True}$$
$$\text{True}$$

b. 3/4 is an integer or 12 > 10.

$$(3/4 \text{ is an integer}) \vee (12 > 10)$$
$$\text{False} \vee \text{True}$$
$$\text{True}$$

c. If 4 < 3 then Carleton University is located on the moon.

$$(4 < 3) \rightarrow (\text{Carleton University is located on the moon})$$
$$\text{False} \rightarrow \text{False}$$
$$\text{True}$$

d. If your instructor's name is Robert then the sky is purple.

$$(\text{Your instructor's name is Robert}) \rightarrow (\text{the sky is purple})$$
$$\text{True} \rightarrow \text{False}$$
$$\text{False}$$

4. Using only the →, ∧, and ¬ operators, find a logical expressions that is equivalent to $p \leftrightarrow (q \vee r)$

$$p \leftrightarrow (q \vee r)$$

by Biconditional Equivalence

$$(p \rightarrow (q \vee r)) \wedge ((q \vee r) \rightarrow p)$$

by DeMorgan's Law

$$(p \rightarrow \neg(\neg q \wedge \neg r)) \wedge (\neg(\neg q \wedge \neg r) \rightarrow p)$$

5. Using only the ∨ and ¬ operators, find a logical expressions that is equivalent to $(p \vee q) \rightarrow r$

$$(p \vee q) \rightarrow r$$

by Implication Equivalence

$$\neg(p \vee q) \vee r$$

6. Determine if the following expressions are tautologies, contradictions, or contingencies by using truth tables. Show all your work.

   a. $((a \vee b) \rightarrow \neg a) \vee b$

a	b	¬a	a ∨ b	(a ∨ b) → ¬a	((a ∨ b) → ¬a) ∨ b
T	T	F	T	F	T
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	F	T	T

   b. $(c \wedge \neg d) \vee ((c \wedge d) \rightarrow c)$

c	d	¬d	c ∧ d	c ∧ ¬d	c ∧ d → c	(c ∧ ¬d) ∨ (c ∧ d → c)
T	T	F	T	F	T	T
T	F	T	F	T	T	T
F	T	F	F	F	T	T
F	F	T	F	F	T	T

c. $\neg((e \rightarrow f) \rightarrow (f \vee \neg e))$

e	f	¬e	e → f	f ∨ ¬e	(e → f) → (f ∨ ¬e)	¬((e → f) → (f ∨ ¬e))
T	T	F	T	T	T	F
T	F	F	F	F	T	F
F	T	T	T	T	T	F
F	F	T	T	T	T	F

d. $((g \vee h) \leftrightarrow ((g \oplus h) \vee (g \rightarrow h)))$

g	h	g ∨ h	g ⊕ h	g → h	(g ⊕ h) ∨ (g → h)	(g ∨ h) ↔ ((g ⊕ h) ∨ (g → h))
T	T	T	F	T	T	T
T	F	T	T	F	T	T
F	T	T	T	T	T	T
F	F	F	F	T	T	F

7. Determine if the following expressions are tautologies, contradictions, or contingencies by using logical equivalences. Show all your work.

a. $((a \vee b) \rightarrow \neg a) \vee b$

$$((a \vee b) \rightarrow \neg a) \vee b$$

Implication Equivalence

$$(\neg(a \vee b) \vee \neg a) \vee b$$

DeMorgan's Law

$$((\neg a \wedge \neg b) \vee \neg a) \vee b$$

Commutativity

$$(\neg a \vee (\neg a \wedge \neg b)) \vee b$$

Absorption

$$\neg a \vee b$$

∴ this is a contingency

b. $(c \wedge \neg d) \vee ((c \wedge d) \rightarrow c)$

$$(c \wedge \neg d) \vee (c \wedge d \rightarrow c)$$

Implication Equivalence

$$(c \wedge \neg d) \vee (\neg(c \wedge d) \vee c)$$

Associativity

$$((c \wedge \neg d) \vee \neg(c \wedge d)) \vee c$$

DeMorgan's Law

$$((c \wedge \neg d) \vee \neg c \vee \neg d) \vee c$$

Commutativity

$$(\neg c \vee (c \wedge \neg d) \vee \neg d) \vee c$$

Distributivity

$$((\neg c \vee c) \wedge (\neg c \vee \neg d)) \vee \neg d) \vee c$$

Negation

$$(T \wedge (\neg c \vee \neg d)) \vee \neg d) \vee c$$

Commutativity

$$((\neg c \vee \neg d) \wedge T) \vee \neg d) \vee c$$

Identity

$$((\neg c \vee \neg d) \vee \neg d) \vee c$$

Associativity

$$(\neg c \vee (\neg d \vee \neg d)) \vee c$$

Idempotence

$$(\neg c \vee \neg d) \vee c$$

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$(\neg c \vee \neg d) \vee c$

Commutativity

$c \vee (\neg c \vee \neg d)$

Associativity

$(c \vee \neg c) \vee \neg d$

Negation

$\text{True} \vee \neg d$

Domination

True

∴ this is a tautology

c. $\neg((e \to f) \to (f \vee \neg e))$

$\neg((e \to f) \to (f \vee \neg e))$

Implication Equivalence

$\neg((\neg e \vee f) \to (f \vee \neg e))$

Implication Equivalence

$\neg(\neg (\neg e \vee f) \vee (f \vee \neg e))$

DeMorgan's Law

$\neg((\neg\neg e \wedge \neg f) \vee (f \vee \neg e))$

Double Negation

$\neg((e \wedge \neg f) \vee (f \vee \neg e))$

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$\neg((e \wedge \neg f) \vee (f \vee \neg e))$

Associativity

$\neg(((e \wedge \neg f) \vee f) \vee \neg e)$

Commutativity

$\neg((f \vee (e \wedge \neg f)) \vee \neg e)$

Distributivity

$\neg(((f \vee e) \wedge (f \vee \neg f)) \vee \neg e)$

Negation

$\neg(((f \vee e) \wedge T) \vee \neg e)$

Identity

$\neg((f \vee e) \vee \neg e)$

Associativity

$\neg(f \vee (e \vee \neg e))$

Negation

$\neg(f \vee T)$

Domination

$\neg T$

F

∴ this is a contradiction

d. $((g \vee h) \leftrightarrow ((g \oplus h) \vee (g \rightarrow h)))$

$$(g \vee h) \leftrightarrow ((g \oplus h) \vee (g \rightarrow h))$$

Implication Equivalence

$$(g \vee h) \leftrightarrow ((g \oplus h) \vee (\neg g \vee h))$$

Exclusive Disjunction Equivalence (from class)

$$(g \vee h) \leftrightarrow (((g \vee h) \wedge \neg(g \wedge h)) \vee (\neg g \vee h))$$

DeMorgan's Law

$$(g \vee h) \leftrightarrow (((g \vee h) \wedge (\neg g \vee \neg h)) \vee (\neg g \vee h))$$

Commutativity

$$(g \vee h) \leftrightarrow ((\neg g \vee h) \vee ((g \vee h) \wedge (\neg g \vee \neg h)))$$

Distributivity

$$(g \vee h) \leftrightarrow (((\neg g \vee h) \vee (g \vee h)) \wedge ((\neg g \vee h) \vee (\neg g \vee \neg h)))$$

n.b., the use of ***red*** *is only to improve visibility at this point*

Associativity

$$(g \vee h) \leftrightarrow ((((\neg g \vee h) \vee g) \vee h) \wedge ((\neg g \vee h) \vee (\neg g \vee \neg h)))$$

Commutativity

$$(g \vee h) \leftrightarrow ((((h \vee \neg g) \vee g) \vee h) \wedge ((\neg g \vee h) \vee (\neg g \vee \neg h)))$$

Associativity

$$(g \vee h) \leftrightarrow (((h \vee (\neg g \vee g)) \vee h) \wedge ((\neg g \vee h) \vee (\neg g \vee \neg h)))$$

Negation

$$(g \vee h) \leftrightarrow (((h \vee T) \vee h) \wedge ((\neg g \vee h) \vee (\neg g \vee \neg h)))$$

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$$(g \lor h) \leftrightarrow (((h \lor T) \lor h) \land ((\neg g \lor h) \lor (\neg g \lor \neg h)))$$

Domination

$$(g \lor h) \leftrightarrow ((T \lor h) \land ((\neg g \lor h) \lor (\neg g \lor \neg h)))$$

Commutativity

$$(g \lor h) \leftrightarrow ((h \lor T) \land ((\neg g \lor h) \lor (\neg g \lor \neg h)))$$

Domination

$$(g \lor h) \leftrightarrow (T \land ((\neg g \lor h) \lor (\neg g \lor \neg h)))$$

Commutatibity

$$(g \lor h) \leftrightarrow (((\neg g \lor h) \lor (\neg g \lor \neg h)) \land T)$$

Identity

$$(g \lor h) \leftrightarrow ((\neg g \lor h) \lor (\neg g \lor \neg h))$$

Commutativity

$$(g \lor h) \leftrightarrow ((h \lor \neg g) \lor (\neg g \lor \neg h))$$

Associativity

$$(g \lor h) \leftrightarrow (((h \lor \neg g) \lor \neg g) \lor \neg h)$$

Associativity

$$(g \lor h) \leftrightarrow ((h \lor (\neg g \lor \neg g)) \lor \neg h)$$

Idempotence

$$(g \lor h) \leftrightarrow ((h \lor \neg g) \lor \neg h)$$

Commutativity

$$(g \lor h) \leftrightarrow ((\neg g \lor h) \lor \neg h)$$

(continued from previous page)

$$(g \vee h) \leftrightarrow ((\neg g \vee h) \vee \neg h)$$

Associativity

$$(g \vee h) \leftrightarrow (\neg g \vee (h \vee \neg h))$$

Negation

$$(g \vee h) \leftrightarrow (\neg g \vee T)$$

Domination

$$(g \vee h) \leftrightarrow T$$

∴ this is a contingency

You can stop here, because, as you know, the biconditional ↔ also means "is equivalent to". It should be obvious that the statement "(g ∨ h) is equivalent to True" is sometimes true, but not always.

8. Let P(x) be the predicate "x is purple", V(x) be the predicate "x is a vegetable", and W(x) be the predicate "x grows in the wild". Translate the following expressions into English. The universe of discourse is all plants.

   a. $\exists a\ (V(a) \vee \neg W(a))$

   There exists a plant that is either a vegetable or does not grow in the wild.

   b. $\forall b\ (W(b) \rightarrow (V(b) \vee P(b)))$

   If a plant grows in the wild then it is either a vegetable or it is purple.

   c. $\exists c\ (P(c) \leftrightarrow W(c))$

   There exists a plant that is purple if and only if it grows in the wild.

9. Negate the following predicate logic statements and write your answer as both an English statement and a predicate logic expression. You must also explicitly state what the atomic predicates are.

Let the universe of discourse be all living things.

a. Every amphibian is green and spotted.

A(x) = x is an amphibian

G(x) = x is green

S(x) = x is spotted

$\forall c \quad A(c) \rightarrow (G(c) \wedge S(c))$

...and now negate this...

$\neg\forall c \quad A(c) \rightarrow (G(c) \wedge S(c))$

$\exists c \quad \neg(A(c) \rightarrow (G(c) \wedge S(c)))$

Implication Equivalence

$\exists c \quad \neg(\neg A(c) \vee (G(c) \wedge S(c)))$

DeMorgan's Law

$\exists c \quad \neg\neg A(c) \wedge (\neg (G(c) \wedge S(c)))$

Double Negation

$\exists c \quad A(c) \wedge (\neg (G(c) \wedge S(c)))$

DeMorgan's Law

$\exists c \quad A(c) \wedge (\neg G(c) \vee \neg S(c))$

There exists a creature on earth that is an amphibian and it is either not green or it is not spotted.

Let the universe of discourse be all living things.

b. There is at least one person in the class who is also in COMP1405.

P(x) = **x is a person in the class**

C(x) = **x is also in COMP1405**

$\exists x \quad P(x) \wedge C(x)$

...and now negate this...

$\neg\exists x \quad P(x) \wedge C(x)$

$\forall x \quad \neg (P(x) \wedge C(x))$

$\forall x \quad \neg P(x) \vee \neg C(x)$

Every living creature on earth is either not a person in this class or is not in COMP1405.