MATH 3007A - Functions of a Complex Variable
Test 1 Solutions
October 6, 2016

[Marks]

[4] 1. Let $z = 2 - 3i$. Determine the real part $Re(z)$, the imaginary part $Im(z)$, the complex conjugate $\overline{z}$, and the magnitude $|z|$, of z.
Solution:
$Re(z) = 2$, $Im(z) = -3$, $\overline{z} = 2 + 3i$, $|z| = \sqrt{13}$.

[2] 2. Express $\dfrac{3-2i}{2+i}$ in the form $a + bi$, $a, b \in \mathbb{R}$.
Solution:
$\dfrac{3-2i}{2+i} = \dfrac{3-2i}{2+i}\dfrac{2-i}{2-i} = \dfrac{4-7i}{5} = \dfrac{4}{5} - \dfrac{7}{5}i$.

[3] 3. Let $z = -1 - \sqrt{3}\,i$.

(a) Express z in polar form.
Solution:
$$|z| = 2 \;\Rightarrow\; z = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2e^{-2\pi i/3} = 2e^{4\pi i/3}.$$

(b) Determine $\arg(z)$ if $\arg(z) \in [0, 2\pi)$.
Solution:
$\arg(z) = \dfrac{4\pi}{3}$.

(c) Determine $\arg(z)$ if $\arg(z) \in [-\pi, \pi)$.
Solution:
$\arg(z) = -\dfrac{2\pi}{3}$.

[2] 4. Let $z = 3e^{-\pi i/3}$.

(a) Determine $\log(z)$ if $\arg(z) \in [0, 2\pi)$.
Solution:
$\log(z) = \ln(3) + \dfrac{5\pi i}{3}$.

(b) Determine $\log(z)$ if $\arg(z) \in [-\pi, \pi)$.
Solution:
$\log(z) = \ln(3) - \dfrac{\pi i}{3}$.

[4] 5. Let $z = -2e^{2\pi i/3}$.

(a) Determine $\log(z)$ if $\arg(z) \in [0, 2\pi)$.
Solution:
$\log(z) = \ln(2) + \dfrac{5\pi i}{3}$.

(b) Determine $\log(z)$ if $\arg(z) \in [-\pi, \pi)$.
Solution:
$\log(z) = \ln(2) - \dfrac{\pi i}{3}$.

[4] 6. Find the square roots of $-i$ and express them in the form $a + bi$, $a, b \in \mathbb{R}$.
Solution:
$-i = e^{\frac{3\pi i}{2}+2\pi ik} \;\Rightarrow\;$ the square roots are $e^{\frac{\pi i}{4}(3+4k)}$, $k = 0, 1$, i.e., $e^{3\pi i/4}$ and $e^{7\pi i/4}$.
$e^{3\pi i/4} = -\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$ and $e^{7\pi i/4} = \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$.

[6] 7. Find the cube roots of 64 and express them in the form $a + bi$, $a, b \in \mathbb{R}$.
Solution:
$64 = 64e^{2\pi ik} \;\Rightarrow\;$ the cube roots are $4e^{2\pi ik/3}$, $k = 0, 1, 2$, i.e., 4, $4e^{2\pi i/3} = -2 + 2\sqrt{3}\,i$, and $4e^{4\pi i/3} = -2 - 2\sqrt{3}\,i$.

[5] 8. Find the fourth roots of $8\sqrt{3} + 8i$. You may leave them in polar form.
Solution:
$8\sqrt{3} + 8i = 16e^{\frac{\pi i}{6}+2\pi ik} \;\Rightarrow\;$ the fourth roots are $2e^{\frac{\pi i}{24}(1+12k)}$, $k = 0, 1, 2, 3$, i.e., $2e^{\pi i/24}$, $2e^{13\pi i/24}$, $2e^{25\pi i/24}$, and $2e^{37\pi i/24}$.