ENGR 213/2
APPLIED ORDINARY DIFFERENTIAL EQUATIONS
Solutions to Midterm Test I   October 4, 2010

(**1**) $y' = e^{3x+2y}, \qquad y(0) = 0$

Re-write as:

$$\frac{dy}{dx} = e^{3x}e^{2y}$$

or

$$e^{-2y}dy = e^{3x}dx$$

The equation is separable.

$\int e^{-2y}dy = \int e^{3x}dx \quad \Rightarrow \quad -\dfrac{e^{-2y}}{2} = \dfrac{1}{3}e^{3x} + c$

From the initial value problem

$y(0) = 0 \quad \Rightarrow \quad e^0 = -\dfrac{2}{3}e^0 - 2c \quad \Rightarrow \quad c = -\dfrac{5}{6}$

The solution is

$$\boxed{e^{-2y} = -\frac{3}{2}e^{3x} + \frac{5}{3}} \tag{1}$$

(**2**) $(\mathbf{1} + \mathbf{y^2}\cos(\mathbf{xy}))\,\mathbf{dx} + (\mathbf{xy}\cos(\mathbf{xy}) + \sin(\mathbf{xy}))\,\mathbf{dy} = \mathbf{0}$

By inspection, this equation is not separable and not linear. It is also not homogenous, or Bernoulli, and cannot be solved by linear substitution $v = Ay + Bx + c$.

To check for exactness, denote $M(x, y) = 1 + y^2\cos(xy)$ and $N(x, y) = xy\cos(xy) + \sin(xy)$. Then

$$\frac{\partial M(x, y)}{\partial y} = 2y\cos(xy) - xy^2\sin(xy)$$

$$\frac{\partial N(x, y)}{\partial x} = y\cos(xy) - xy^2\sin(xy) + y\cos(xy)$$

As the above expressions are equal, the equation is exact. We look now for a function $f(x, y)$ such that

$$\frac{\partial f(x, y)}{\partial x} = M(x, y), \quad \Longrightarrow \quad f(x, y) = \int \Big(1 + y^2\cos(xy)\Big)\,\partial x + h(y) = x + y\sin(xy) + h(y)$$

and

$$\frac{\partial f(x,y)}{\partial y} = N(x,y)\,(\equiv xy\cos(xy) + \sin(xy)) = xy\cos(xy) + \sin(xy) + h(y),$$

implying that $h'(y) = 0$, or $h(y) = const$. The solution is

$$\boxed{f(x,y) = x + y\sin(xy) = c} \tag{2}$$

(3) $\mathbf{(x^2 + y^2\sqrt{x^2+y^2})dx = xy\sqrt{x^2+y^2}\,dy, \quad y(1) = 1}$

Again, by inspection, this equation is not separable and not linear. It is not Bernoulli and cannot be solved by linear substitution $v = Ay + Bx + c$. Is it exact? Set $M(x,y) = x^2 + y^2\sqrt{x^2+y^2}$ and $N(x,y) = xy\sqrt{x^2+y^2}$ Then

$$\frac{\partial M(x,y)}{\partial y} = 2y\sqrt{x^2+y^2} + \frac{y^3}{\sqrt{x^2+y^2}}$$

$$\frac{\partial N(x,y)}{\partial x} = y\sqrt{x^2+y^2} + \frac{x^2 y}{\sqrt{x^2+y^2}}$$

So the equation is not exact. It is not homogeneous. However, it is is simplified by noting that

$$\frac{dy}{dx} = \frac{x^2 + y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}}$$

which becomes, setting $v = \dfrac{y}{x}, \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\frac{dy}{dx} = \frac{x^2 + v^2x^3\sqrt{1+v^2}}{vx^3\sqrt{1+v^2}} = v + x\frac{dv}{dx}$$

or

$$x\frac{dv}{dx} = \frac{1}{vx\sqrt{1+v^2}}$$

which is separable. We get

$$\int v\sqrt{1+v^2}dv = \int \frac{dx}{x^2}$$

Setting $u = 1 + v^2, \quad 2vdv = du$, the equation becomes

$$\int \frac{1}{2}u^{\frac{1}{2}}du = \int \frac{1}{x^2}dx$$

integrating we get

$$\frac{1}{3}u^{\frac{3}{2}} = -\frac{1}{x} + c$$

or

$$\left(1 + \frac{y^2}{x^2}\right)^3 = 9\left(-\frac{1}{x} + c\right)^2$$

Imposing the IVP $y(1) = 1$, $\Longrightarrow 8 = 9(-1+c)^2$, $\Longrightarrow c = \frac{\sqrt{8}}{3} + 1$. The final solution becomes

$$\boxed{y^2 = x^2\left[\left(\sqrt{8} + 3 - \frac{3}{x}\right)^{\frac{2}{3}} - 1\right]} \qquad (3)$$

(4) $\mathbf{2xdy - (y - 2xy^3 \cos x)dx = 0}$

By inspection, this equation is not separable, not linear and not exact. It is also not homogeneous and cannot be solved by linear substitution $v = Ay + Bx + c$. Write as

$$\frac{dy}{dx} - \frac{y}{2x} = -y^3 \cos x$$

So it is Bernoulli, with $p = 3$. Making the change of variable $v = y^{1-3} = \frac{1}{y^2}$, the equation becomes a linear equation in v:

$$\frac{dv}{dx} - (1-3)\frac{v}{2x} = -(1-3)\cos x$$

We solve it by evaluating first the integrating factor

$$\mu(x) = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

and then

$$v(x) = \frac{1}{x}\left[\int (2x\cos x)dx + C\right] = \frac{2x\sin x}{x} + \frac{2\cos x}{x} + \frac{C}{x}$$

or

$$\boxed{y^2 = \frac{x}{C + 2\left(x\sin x + \cos x\right)}} \qquad (4)$$