

ENGR-213, Sample test 1

For all equations, determine their type and solve them. In Problem 1, solve the Initial Value Problem.

1. $(1 - x^2)y' + 2xy = x^4 - 2x^2 + 1; \quad y(0) = 2.$

2. $y' = \frac{e^{2x+y}}{e^y - 1};$

3. $\frac{2x+y}{x^2+xy}dx + \frac{x+2y}{xy+y^2}dy = 0;$

4. $x^3y' - 2x^2y = x^4y^3;$

5. The mass of a radioactive element on January 1, 1900, was 1g. On January 1, 2000, its mass was 0.99g. Find the half-time $T_{1/2}$ for this element.

Solutions

1. This is a linear equation.

$$y' + \frac{2x}{1-x^2}y = \frac{x^4 - 2x^2 + 1}{1-x^2} = 1 - x^2;$$

Integrating factor $M = e^{\int \frac{2x}{1-x^2}dx} = e^{-\ln|1-x^2|} = \frac{1}{1-x^2};$

$$\frac{1}{1-x^2}y' + \frac{2x}{(1-x^2)^2}y = 1;$$

$$\frac{d}{dx}\left(\frac{1}{1-x^2}y\right) = 1;$$

$$\frac{1}{1-x^2}y = x + C;$$

$$y = x(1 - x^2) + C(1 - x^2);$$

$$y(0) = C; \quad C = 2;$$

Answer: $y = x(1 - x^2) + 2(1 - x^2)$.

2. $y' = \frac{e^{2x+y}}{e^y-1} = e^{2x} \cdot \frac{e^y}{e^y-1}$; this is a separable equation.

$$\frac{e^y - 1}{e^y} dy = e^{2x} dx;$$

$$(1 - e^{-y}) dy = e^{2x} dx;$$

$$\int (1 - e^{-y}) dy = \int e^{2x} dx + C;$$

$$y + e^{-y} = \frac{1}{2} e^{2x} + C;$$

$$2(y + e^{-y}) + C = e^{2x};$$

Answer: $x = \frac{1}{2} \ln(2y + 2e^{-y} + C)$.

3.

$$\frac{2x + y}{x^2 + xy} dx + \frac{x + 2y}{xy + y^2} dy = 0.$$

Let us see if this is an exact equation. Here

$$M = \frac{2x + y}{x^2 + xy}, \quad N = \frac{x + 2y}{xy + y^2};$$

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{(x^2 + xy) - (2x + y)x}{x^2(x + y)^2} \\ &= \frac{x^2 + xy - 2x^2 - xy}{x^2(x + y)^2} = \frac{-x^2}{x^2(x + y)^2} = -\frac{1}{(x + y)^2}.\end{aligned}$$

$$\frac{\partial N}{\partial x} = \frac{y(x + y) - (x + 2y) \cdot y}{y^2(x + y)^2} = \frac{-y^2}{y^2(x + y)^2} = -\frac{1}{(x + y)^2}.$$

$\partial M/\partial y = \partial N/\partial x$; the equation is exact. Let us find the potential $f(x, y)$ satisfying $\partial f/\partial x = M$, $\partial f/\partial y = N$.

$$f(x, y) = \int \frac{2x + y}{x^2 + xy} dx + g(y) = \int \left(\frac{a}{x} + \frac{b}{x + y}\right) dx + g(y);$$

$$ax + ay + bx = (a + b)x + ay = 2x + y; a = 1, b = 1.$$

$$f(x, y) = \int \left(\frac{1}{x} + \frac{1}{x + y}\right) dx + g(y) = \ln|x| + \ln|x + y| + g(y);$$

$$\frac{\partial f}{\partial y} = \frac{1}{x + y} + g'(y) = \frac{x + 2y}{xy + y^2} = \frac{1}{x + y} + \frac{1}{y};$$

$$g'(y) = \frac{1}{y}; g(y) = \ln|y + C|; f(x, y) = \ln|x| + \ln|y| + \ln|x + y|.$$

Answer: $\ln|x| + \ln|y| + \ln|x + y| = C$.

4. $x^3y' - 2x^2y = x^4y^3$; This is a Bernoulli equation.

$$y' - \frac{2}{x}y = xy^3;$$

Let us make a substitution $y = z^p$; $y' = pz^{p-1}z'$;

$$pz^{p-1}z' - \frac{2}{x}z^p = xz^{3p};$$

$$z' - \frac{2}{px}z = \frac{1}{p}xz^{2p+1};$$

Choose $p = -\frac{1}{2}$; the equation becomes linear.

$$z' + \frac{4}{x}z = -2x;$$

Integrating factor is $e^{\int \frac{4}{x}dx} = e^{4\ln|x|} = x^4$;

$$x^4z' + 4x^3z = -2x^5;$$

$$\frac{d}{dx}(x^4z) = -2x^5;$$

$$x^4z = -\frac{2}{6}x^6 + C;$$

$$z = -\frac{1}{3}x^2 + \frac{C}{x^4};$$

$$y = z^{-1/2} = \left(\frac{C}{x^4} - \frac{1}{3}x^2\right)^{-1/2}.$$

Answer: $y = \left(\frac{C}{x^4} - \frac{1}{3}x^2\right)^{-1/2}$.

5. Let k be the decay rate (measured in $year^{-1}$). Then

$$\frac{m(t)}{m(0)} = e^{-kt}$$

where t is measured in years. In our case, $t = 100$, and $e^{-100k} = 0.99$; thus, $-100k = \ln(0.99) = -0.01$, and $k = 10^{-4} \text{year}^{-1}$. The half-time $T_{1/2} = \ln 2/k = 0.6931/10^{-4} = 6931(\text{years})$.

Sample test 2

1. $y' + y + \frac{e^x}{y^2} = 0$;
2. $(e^x \cos y - 2e^{-2y} \cos x)dx + (-e^x \sin y + 2e^{-2y} \sin 2x)dy = 0$;
3. $\frac{dy}{dx} = \frac{xy+x-2y-2}{xy-3x+y-3}$, $y(1) = 0$.
4. $x \frac{dy}{dx} + (x+1)y = e^x$;

5. Radioactive element has a half-time $T_{1/2} = 100 \text{sec}$. At the initial moment there were 10^6 atoms of this element. How long should we wait until 10^4 atoms remain?

Solutions

1. $y' + y + \frac{e^x}{y^2} = 0$. This is a Bernoulli equation. Substitution $y = z^p$; $y' = pz^{p-1}z'$;

$$pz^{p-1}z' + z^p + e^x z^{-2p} = 0;$$

$$z' + \frac{1}{p}z + \frac{e^x}{p}x^{-3p+1} = 0;$$

If we choose $p = \frac{1}{3}$, we get a linear equation

$$z' + 3z + 3e^x = 0.$$

Integrating factor is $e^{\int 3dx} = e^{3x}$;

$$e^{3x}(z' + 3z) + 3e^{4x} = 0;$$

$$\frac{d}{dx}(e^{3x}z) + 3e^{4x} = 0;$$

$$e^{3x}z + \frac{3}{4}e^{4x} = C;$$

$$z = Ce^{-3x} - \frac{3}{4}e^x;$$

Answer: $y = (Ce^{-3x} - \frac{3}{4}e^x)^{1/3}$.

2. $(e^x \cos y - 2e^{-2y} \cos x)dx + (-e^x \sin y + 2e^{-2y} \sin 2x)dy = 0$; let's see if this equation is exact. Here $M = e^x \cos y - 2e^{-2y} \cos x$, $N = -e^x \sin y + 2e^{-2y} \sin 2x$;

$$\frac{\partial M}{\partial y} = -e^x \sin y + 4e^{-2y} \cos 2x,$$

$$\frac{\partial N}{\partial x} = -e^x \sin y + 4e^{-2y} \cos 2x = \frac{\partial M}{\partial y}.$$

The equation is exact, and the solution has the form $f(x, y) = C$ where $f(x, y)$ is a function such that $\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$.

$$\frac{\partial f}{\partial x} = e^x \cos y - 2e^{-2y} \cos 2x;$$

$$f = \int (e^x \cos y - 2e^{-2y} \cos 2x)dx$$

$$= e^x \cos y - e^{-2y} \sin x + g(y);$$

$$\frac{\partial f}{\partial y} = -e^x \sin y + 2e^{-2y} \sin 2x + g'(y)$$

$$= -e^x \sin y + 2e^{-2y} \sin 2x; \quad g'(y) = 0; \quad g(y) = C.$$

Answer: the solution has the form $e^x \cos y - 2e^{-2y} \sin 2x = C$.

3.

$$\begin{aligned} \frac{dy}{dx} &= \frac{xy + x - 2y - 2}{xy - 3x + y - 3} = \frac{y(x-2) + (x-2)}{x(y-3) + (y-3)} \\ &= \frac{(y+1)(x-2)}{(x+1)(y-3)} = \left(\frac{x-2}{x+1}\right) \left(\frac{y+1}{y-3}\right); \end{aligned}$$

The equation is separable:

$$\begin{aligned} \left(\frac{y-3}{y+1}\right) dy &= \left(\frac{x-2}{x+1}\right) dx; \\ \int \left(1 - \frac{4}{y+1}\right) dy &= \int \left(1 - \frac{3}{x+1}\right) dx; \end{aligned}$$

$$y - 4 \ln |y+1| = x - 3 \ln |x+1| + C; \quad \frac{e^y}{(y+1)^4} = C \frac{e^x}{(x+1)^3};$$

To satisfy the initial condition, substitute $x = 1, y = 0$:

$$1 = C \frac{e^1}{2^3} = C \frac{e}{8}; \quad C = \frac{8}{e}.$$

Answer:

$$\frac{e^y}{(y+1)^4} = \frac{8e^{x-1}}{|x+1|^3}.$$

4. $x \frac{dy}{dx} + (x+1)y = e^x$; this is a linear nonhomogeneous equation. Find the integrating factor.

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{e^x}{x};$$

The integrating factor

$$M = e^{\int(1+1/x)dx} = e^x \cdot x;$$

$$xe^x \frac{dy}{dx} + (x+1)e^x y = xe^{2x}; \quad \frac{d}{dx}(xe^x y) = xe^{2x};$$

$$xe^x y = \int xe^{2x} dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C;$$

Answer: $y = \frac{1}{2}e^x - \frac{1}{4} \frac{e^x}{x} + \frac{C}{xe^x}.$

5. If k is the rate of decay, then $T_{1/2} = 100 = \frac{\ln 2}{k}$ (time is measured in seconds). So, $k = \frac{\ln 2}{100} = 0.00693$. The number of the atoms $N(t)$ satisfies the equation $\frac{dN}{dt} = -kN$, and hence $N(t) = N(0)e^{-kt}$. In our problem, $N(t) = 10^{-2} \cdot N(0)$, and hence $e^{-kt} = 10^{-2}$; $-kt = -2 \ln 10$; $t = 2 \ln 10/k = 664.5(sec)$.

Answer: $t = 664.5sec.$