MATH 1009 - B   TEST 4   March 2016   SOLUTIONS

1. [12 Marks] Find f_x, f_y, f_{xx} and f_{yy} of the following functions:

[4] **(a)** $f(x,y) = 2x^3 - y^5 + xy^3$. [4] **(b)** $f(x,y) = e^{2xy}$. [4] **(c)** $f(x,y) = \dfrac{3x}{y}$.

Solution:

(a) $f_x = 6x^2 + y^3$; $f_y = -5y^4 + 3xy^2$; $f_{yy} = -20y^3 + 6xy$; $f_{xx} = 12x$;

(b) $f_x = e^{2xy} \cdot 2y$; $f_y = e^{2xy} \cdot 2x$;

$f_{xx} = 2ye^{2xy} \cdot 2y = 4y^2e^{2xy}$; $f_{yy} = 2xe^{2xy} \cdot 2x = 4x^2e^{2xy}$;

(c) $f_x = \dfrac{\partial}{\partial x}\left(3xy^{-1}\right) = 3y^{-1}$; $f_y = \dfrac{\partial}{\partial x}\left(3xy^{-1}\right) = 3x \cdot (-1)y^{-2} = -\dfrac{3x}{y^2}$;

$f_{xx} = 0$; $f_{yy} = -3x \cdot (-2)y^{-3} = \dfrac{6x}{y^3}$.

2. [12 Marks] If exactly 60 people sign for an aerobics club membership, the price is \$120 per person per year. If more people join, then the price is reduced by \$1 for each additional person. Determine how many members will result in a maximum revenue for the club. Find the optimal price of membership.

Solution:

Let x be the number of additional members. Then the total number of club members is $(60 + x)$, and the new price for the membership is

$$p(x) = 120 - 1 \cdot x = 120 - x \ \text{(dollars)}.$$

Thus, the revenue of the club to be maximized is given by the product of the number of members and the price of membership, that is,

$$R(x) = (60 + x)(120 - x) = 7,200 + 60x - x^2.$$

Since the price $p(x) = 120 - x$ must be a nonnegative number, the restriction $120 - x \geq 0$ requires $x \leq 120$. Also, as x denotes the number of additional members of the club, $x \geq 0$. Thus, we have to maximize $P(x)$ over the domain $0 \leq x \leq 120$.

Compute the first derivative of $R(x)$ and find all the critical numbers in the domain.

$$R'(x) = 60 - 2x = 0 \ \text{when} \ 2x = 60, \ \Rightarrow \ x = 30.$$

The critical number $x = 30$ is in the domain. Since we have to find the absolute maximum of $R(x)$ on the closed interval $[0, 120]$, we compute

$$R(0) = 7,200; \quad R(30) = 8,100; \quad R(120) = 0.$$

Thus, the largest value is $R(30) = 8,100$, so the maximum revenue is obtained when the club has 30 extra members signed up. Then the optimal price of membership is $p = 60 + 30 = 90$ dollars.

3. [16 Marks] Consider the function $f(x) = x^4 - 2x^2 + 1$.

[1] **(a)** State the domain of f: $Dom f = (-\infty, \infty)$.

[2] **(b)** Find the y-intercept. Find the asymptotes, if any.

$y = f(0) = 1$; No asymptotes.

[6] **(c)** Determine the intervals where f is increasing and those where f is decreasing. Find the relative minima and maxima of f and the value of f at these points.

$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0 \Rightarrow x = 0$ and $x = \pm 1$ are the critical numbers of f.

The function is increasing when $f'(x) = 4x(x^2 - 1) > 0$, i.e. when $x \in (-1, 0) \bigcup (1, \infty)$.

The function is decreasing when $f'(x) = 4x(x^2 - 1) < 0$, i.e. when $x \in (-\infty, -1) \bigcup (0, 1)$.

Since $f'(x)$ is changing its sign from " + " to " − " while passing across $x = 0$, then there is a local maximum at $x = 0$. $f(0) = 1$.

Since $f'(x)$ is changing its sign from " − " to " + " while passing across $x = \pm 1$, then there is a local minimum at $x = -1$ and $x = 1$. $f(-1) = f(1) = 0$.

[6] **(d)** Determine the intervals where the graph of the function is concave up and those where it is concave down. Find the inflection points of f. (Give both the x- and the y-coordinates).

$$f''(x) = 12x^2 - 4 = 0 \text{ when } x^2 = \frac{4}{12} = \frac{1}{3}, \quad x = \pm\sqrt{\frac{1}{3}} = \pm\frac{1}{\sqrt{3}}.$$

$f''(x) > 0$ on $(-\infty, -\frac{1}{\sqrt{3}})$ and $(\frac{1}{\sqrt{3}}, \infty)$, so the graph is concave UP.

$f''(x) < 0$ on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ so the graph is concave DOWN.

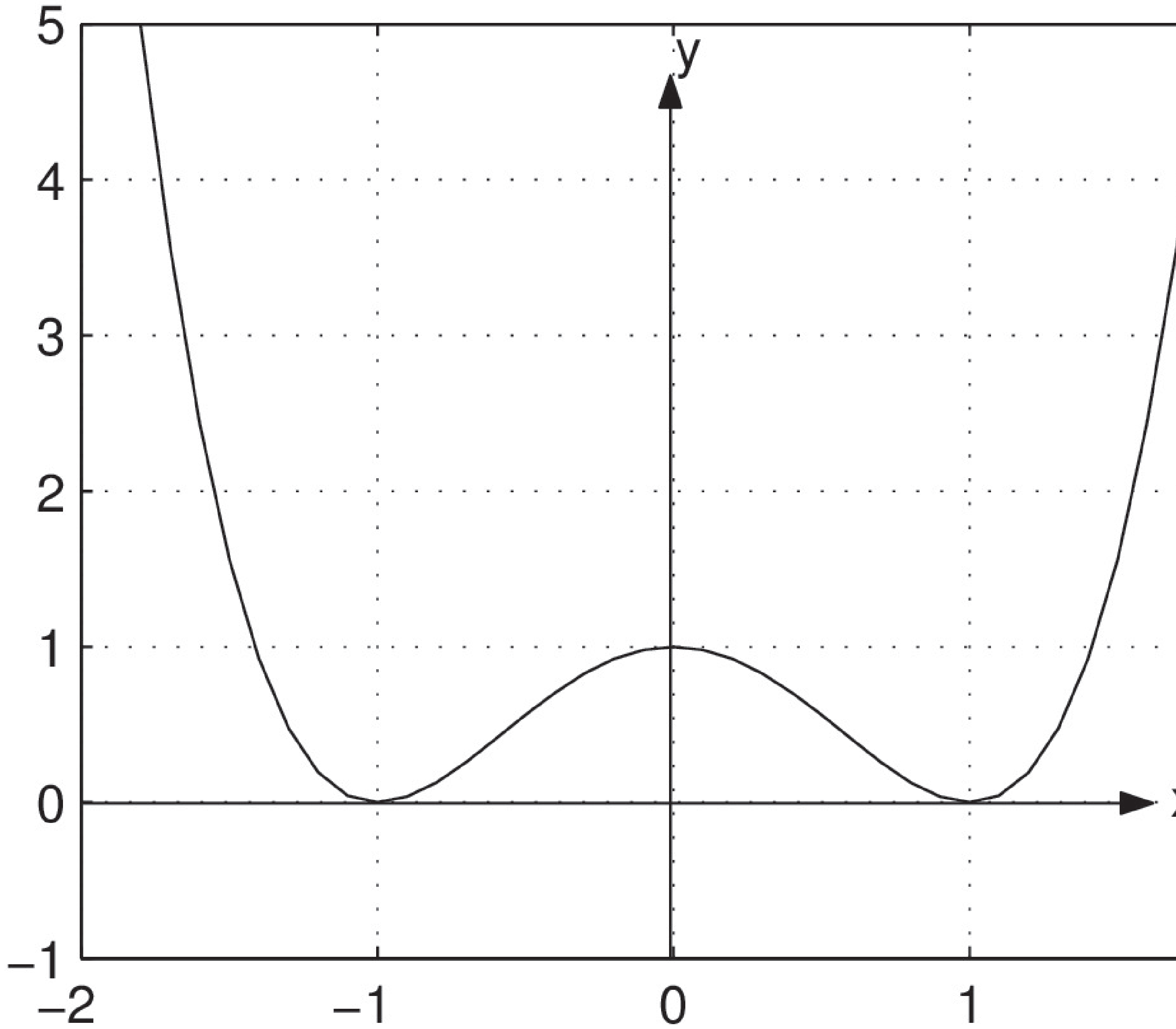
The inflection points are $\left(-\frac{1}{\sqrt{3}}, f(-\frac{1}{\sqrt{3}})\right) = \left(-\frac{1}{\sqrt{3}}, \frac{4}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, f(\frac{1}{\sqrt{3}})\right) = \left(\frac{1}{\sqrt{3}}, \frac{4}{9}\right)$.

[1] **(e)** Sketch the graph of f.

(d) The graph:

Marking guidelines:

Problem 2: 6 marks for the revenue function, 2 marks for the critical number $x = 30$, 1 mark for the absolute max (comparing the revenue), 1 mark for the optimal price.



Graph of $f(x) = x^4 - 2x^2 + 1$