School of Mathematics and Statistics
Carleton University
Math. 1004C, Fall 2016

TEST 2

Non-programmable calculators permitted as well as a few blank sheets but these should NOT be submitted.
Print Name :

Student Number:

Tutorial Section (C1, C2, C3, C4, or C5):

PART I: Multiple Choice Questions

(Choose and CIRCLE only ONE answer - No part marks here.)

1. **[2 marks]** Calculate the derivative $f'(\pi/4)$ of f where $f(x) = \text{Arctan}(\sin x)$.
   Solution: $f'(x) = \dfrac{\cos x}{\sin^2 x + 1}$, $f'(\pi/4) = \dfrac{1/\sqrt{2}}{(1/\sqrt{2})^2 + 1} = \dfrac{\sqrt{2}}{3}$.
   (a) $\sqrt{3}/2$, (b) 0, $\boxed{\text{(c) } \sqrt{2}/3}$, (d) $-1/\sqrt{2}$, e) none of these

2. **[2 marks]** Let f be a differentiable function with $f'(1) = 3$ whose differentiable inverse, F, satisfies $F(-2) = 1$. Find the value of $F'(-2)$.
   Solution: $F'(-2) = \dfrac{1}{f'(F(-2))} = \dfrac{1}{f'(1)} = \dfrac{1}{3}$.
   (a) 1, (b) $-1/2$, $\boxed{\text{(c) } 1/3}$, (d) -3, e) none of these

3. **[2 marks]** Evaluate the following limit: $\displaystyle\lim_{x\to\pi/2} \frac{\sec x}{\tan^2 x}$.
   Solution: We have an indeterm. form $\left[\dfrac{\infty}{\infty}\right]$, the func. are differentiable, so by L'Hospital's rule:
   $\displaystyle\lim_{x\to\pi/2} \frac{\sec x \tan x}{2\tan x \sec^2 x} = \frac{1}{2}\lim_{x\to\pi/2} \frac{1}{\sec x} = \frac{1}{2}\lim_{x\to\pi/2} \cos x = 0.$
   OR just by simplification:
   $\displaystyle\lim_{x\to\pi/2} \frac{\sec x}{\tan^2 x} = \lim_{x\to\pi/2} \frac{\cos x}{\sin^2 x} = 0$
   (a) $-1/2$, $\boxed{\text{(b) } 0}$, (c) 1, (d) ∞, e) The limit does not exist

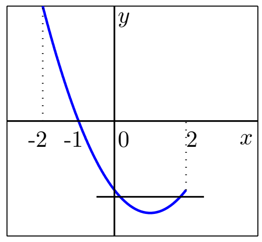
4. **[2 marks]** Let $f(x) = \cos(\text{Arctan}(x^2))$, compute $f'(x)$ at the point $x = 1$
   Solution: $\left(\cos(\text{Arctan}(x^2))\right)' = -\sin(\text{Arctan}(x^2)) \cdot \dfrac{1}{1+x^4} \cdot 2x$,
   at $x = 1$: $-\sin(\text{Arctan}(1)) \cdot \dfrac{1}{1+1} \cdot 2 = -\sin(\pi/4) = -\dfrac{1}{\sqrt{2}}$
   (a) $\sqrt{2}$, (b) 0, (c) 1, $\boxed{\text{(d) } -1/\sqrt{2}}$, e) none of these

5. **[2 marks]** Let $y^2 = \sin(x - y)$ define a differentiable curve in the $xy-$plane and that y may also be defined as a differentiable function of x near $x = \pi$. Then the slope of the tangent line to this curve at the point $x = \pi, y = 0$ is equal to 1.
   Solution: $2yy' = \cos(x-y) \cdot (1 - y')$ from here $y' = \dfrac{\cos(x-y)}{2y + \cos(x-y)}$.
   The slope at $(\pi, 0)$: $m = y'(\pi, 0) = \dfrac{\cos\pi}{\cos\pi} = 1$
   $\boxed{\text{(a) TRUE}}$, (b) FALSE,

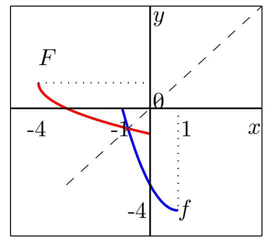
PART II: Show all work here and give details. No additional pages will be accepted

6. **[3+4+3 marks]** Let $f(x) = (x+1)(x-3)$.
   a) Let the domain of f be the interval $[-2, 2]$. Does f have an inverse function? Explain.
   b) Show that f has an inverse function, F, if its domain is the interval $[1, -1]$. What is the domain and range of F?
   c) Calculate the derivative $F'(x)$ of the previous inverse function, F.

Solution:

a) 

the function fails the horizontal line test: it is not one-to-one on $[-2,2]$

b) 

the function is one-to-one on $[-1,1]$

Dom(f)=Ran(F)=$[-1,1]$

Ran(f)=Dom(F)=$[-4,0]$

c) $F'(x) = \dfrac{1}{f'(F(x))} = \dfrac{1}{((y+1)(y-3))'} = \dfrac{1}{2y-2}$
$(f\circ F)(x) \equiv f(F(x)) = (y+1)(y-3) = x$ it follows $F(x) \equiv y = 1-\sqrt{4+x}$ (we should take the lower branch), and hence $F'(x) = \dfrac{1}{-2\sqrt{4+x}}$
OR
as $F(x) = 1-\sqrt{4+x}$ then directly $F'(x) = -\dfrac{1}{2\sqrt{4+x}}$.

7. **[5+5 marks]** a) Evaluate the following limit using any method: $\lim\limits_{x\to 0} \dfrac{x-\tan x}{x^2}$.

Solution: It's an indeterminate form $\left[\dfrac{0}{0}\right]$, the functions are differentiable, so by L'Hospital's rule:
$\lim\limits_{x\to 0}\dfrac{1-\sec^2 x}{2x} = \lim\limits_{x\to 0}(1+\sec x)\cdot\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{1-\sec x}{x} = \left[\dfrac{0}{0}\right] = \lim\limits_{x\to 0}\dfrac{-\sec x\tan x}{1} = 0.$

b) Evaluate the following limit using any method: $\lim\limits_{x\to 5}\dfrac{\sqrt{4+x}-3}{x-5}$.

Solution: It's an indeterminate form $\left[\dfrac{0}{0}\right]$, the functions are differentiable, so by L'Hospital's rule:
$\lim\limits_{x\to 5}\dfrac{\sqrt{4+x}-3}{x-5} = \lim\limits_{x\to 5}\dfrac{\frac{1}{2\sqrt{4+x}}}{1} = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$;
OR
$\lim\limits_{x\to 5}\dfrac{(\sqrt{4+x}-3)(\sqrt{4+x}+3)}{(x-5)(\sqrt{4+x}+3)} = \lim\limits_{x\to 5}\dfrac{(4+x-9)}{(x-5)(\sqrt{4+x}+3)} = \dfrac{1}{\sqrt{4+5}+3} = \dfrac{1}{6}.$