MATH 235 COURSE SUMMARY

Linear Transformations [Sec. 4.1 – 4.5]

- Definitions:
  - Linear Transformation: $L \textit{ linear iff } \begin{cases} L(v_1 + v_2) = L(v_1) + L(v_2) \\ L(cv_1) = c \cdot L(v_1) \end{cases}$
  - Kernel: $\ker(L) = \{v \mid L(v) = 0, v \in V\}$
  - Range: $range(L) = \{L(v) \mid v \in V\}$
- Dimension Theorem: $\dim(\ker L) + \dim(rangeL) = \dim(V)$
  or $nullity(L) + rank(L) = \dim(V)$
- Definitions:
  - One-to-One: $L : V \to W \textit{ one} - \textit{to} - \textit{one iff } v_1 \neq v_2 \Rightarrow L(v_1) \neq L(v_2)$
  - Onto: $L : V \to W \textit{ onto iff } range(L) = W$

  $\textit{Iff } \dim(V) = \dim(W), \quad L \textit{ one} - \textit{to} - \textit{one} \Leftrightarrow L \textit{ onto}$
- Matrix Form of a Linear Transformation: $[L(\vec{x})]_\beta = [L]_\beta^\alpha \cdot [\vec{x}]_\alpha$
- Change of Basis Matrices: $[x]_\beta = [I]_\beta^\alpha \cdot [x]_\alpha$
- Composition of Transformations: $L_1 \circ L_2(v) = L_1(L_2(v))$
- Similarity: $A \sim B \quad \textit{iff} \quad A = PBP^{-1}$
- Invertibility: $L \textit{ invertible iff } [L] \textit{ invertible} \;\&\; [L^{-1}] = [L]^{-1}$
- Isomorphisms: $V \textit{ is isomorphic to } W \textit{ iff } \dim(V) = \dim(W)$

Determinants [Sec. 5.3 – 5.4]

- Determinant of a 2X2 Matrix: $\det(A) = a_{11}a_{22} - a_{12}a_{21}$
- Determinant of Larger Matrices: $\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$
- Triangular Matrix Determinants: $\textit{If } A \textit{ is triangular}, \quad \det(A) = \prod_{i=1}^{n} a_{ii}$
- Row Operations:
  - switching two rows of A: $\det(B) = -\det(A)$
  - multiplying a row of A by k: $\det(B) = k\det(A)$
  - adding two rows of A: $\det(B) = \det(A)$
- Properties of the Determinant: $\det(A^T) = \det(A)$ $\qquad \det(A^{-1}) = (\det(A))^{-1}$

  $\det(AB) = \det(A) \cdot \det(B)$ $\qquad \det(AB) = \det(BA)$
- Invertibility: $A \textit{ is invertible iff } \det(A) \neq 0$
- Similarity: $\textit{If } A \sim B, \textit{ then } \det(A) = \det(B)$
- Adjugate/Adjunct: $(adj(A))_{ij} = C_{ji}(A)$
- Inverse Formula: $A^{-1} = \left(\frac{1}{\det(A)}\right) \cdot adj(A)$

Invariance

- Restrictions: *For* $L: V \to W$ & $U \subseteq V$, $L|_U : U \to W$ $L|_U\ (u) = L(u)$
- Invariance: *For* $L: V \to V$ & $U \subseteq V$, U *is L-invariant iff* $L(U) \subseteq U$
- Cyclic Subspaces: $L-$*cyclic subspace of* V *by* v *is* $S_v = span(\{v, L(v), L^2(v), ...\})$

Eigenspaces & Root Spaces [Sec. 6.1, notes]

- Eigenvectors, Eigenvalues of L: $L(v) = \lambda v$ or $[L][v] = \lambda[v]$
  - Eigenpair of L: (λ, v)
- Minimal Polynomial "with respect to v":

  Find $v, Av, A^2v, ..., A^k v$ such that $A^k v = c_0 v + c_1 Av + c_2 A^2 v + ... + c_{k-1}A^{k-1}v$ and then:

  $$p_v(t) = t^k - c_{k-1}t^{k-1} - ... - c_2 t^2 - c_1 t - c_0$$
- Characteristic Polynomial "of L": $\Delta_{[L]}(t) = \det(tI - [L])$

  $\Delta_{[L]}(t) = (t-\lambda_1)^{r1}(t-\lambda_2)^{r2}...$
- Minimal Polynomial "of L":
  - "lowest degree monic polynomial *p(t)* for which *p(A)v = 0* for all *v*"
  - look at $\Delta_{[L]}(t)$ and start with $r_1 = r_2 = ... = 1$ and go up to try to get *p(A)v = 0*
- Solving the Matrix Eigenvalue Equation:
  - solve the polynomial $\Delta_{[L]}(\lambda) = 0$ for all λ, then solve $(\lambda I - [L])v = 0$ for all v
- Eigenspace (corresponding to λ): $E_\lambda = \{v \mid L(v) = \lambda v, v \in V\}$ and $E_\lambda \subseteq V$

  $E_\lambda = \ker(\lambda I - [L])$
- Geometric Multiplicity (of λ): $g_\lambda = \dim(E_\lambda)$
- Root Vectors: $(\lambda I - [L])^r v = 0$ for some r
  - Index: index of v is the smallest value of r above
- Root Space (corresponding to λ): $R_\lambda = \ker(\lambda I - [L])^n$ for $n = \dim(V)$
- Algebraic Multiplicity (of λ): $a_\lambda = \dim(R_\lambda)$

  $index(R_\lambda),\ g_\lambda \le a_\lambda$

Diagonalizability & Orthogonality [Sec. 6.2, 6.4, notes]

- Diagonalizable: $L: V \to W$ *is diagonalizable iff V has a basis* β *of eigenvectors*

  Further, $$[L]_\beta^\beta = \begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & \lambda_n \end{bmatrix}$$

- Complex Inner Product: $\langle x, y\rangle = x^T\ \overline{y}$
- Properties of Complex Inner Product:

$$\langle x, y_1 + y_2\rangle = \langle x, y_1\rangle + \langle x, y_2\rangle$$
$$\langle x_1 + x_2, y\rangle = \langle x_1, y\rangle + \langle x_2, y\rangle$$
$$\langle cx, y\rangle = c\langle x, y\rangle$$
$$\langle x, cy\rangle = \overline{c}\langle x, y\rangle$$
$$\langle y, x\rangle = \overline{\langle x, y\rangle}$$

- Orthogonality: x & y *orthogonal iff* $\langle x, y\rangle = 0$
- Orthogonal Compliment: $U^{\perp} =$ *the set of all vectors orthogonal to all vectors in* U

$$U^{\perp} = \bigcap_{u \in U} u^{\perp} \quad \text{such that} \quad u^T = \{y \mid \langle y, u\rangle = 0\}$$

- Adjoint (Conjugate Transpose): $A^* = \overline{A}^T$

$$\langle x, Ay\rangle = \langle A^* x, y\rangle$$

  - for real matrices: $A^* = A^T$
- Hermitian Matrices: *A is Hermitian iff* $A^* = A$
  - for real matrices: *A is Hermitian* $\Leftrightarrow$ *A is [real] symmetric*
- Skew Hermitian Matrices: *A is skew Hermitian iff* $A^* = -A$
  - for real matrices: *A is skew Hermitian* $\Leftrightarrow$ *A is [real] skew symmetric*
- Invariance: *U is A-invariant iff $U^{\perp}$ is A^*-invariant*
- Self-Adjoint *A is self-adjoint iff* $\langle x, Ay\rangle = \langle Ax, y\rangle$

  ie- Hermitian operators are self-adjoint w.r.t. complex inner product
- Unitary Matrices: *A is unitary iff* $A^* A = I_n$
  - A is then said to "preserve length"
  - the columns of A form an orthonormal basis for C^n
  - for real matrices: *A is unitary* $\Leftrightarrow$ *A is orthogonal*
- Properties of Hermitian:
  - Hermitian matrices have real eigenvalues.
  - Each eigenspace is orthogonal to one another.
  - Hermitian matrices are diagonalizable
  - The diagonal matrix is $L^* HL$ where L is unitary
- Triangulation:

  If $A \in M_{n\times n}(C)$ *then* $L^{-1}AL$ *is lower triangular where L is some unitary matrix.*

-- Ryan Newson, W2002