MAT 1341E Final Exam

Fall 2016

December 13

Professor: Charles Starling

Family Name:______________________________

First Name:______________________________

Student Number:______________________________

Enter Multiple Choice Answers Here	
1	E
2	D
3	C
4	D
5	D
6	E
7	A
8	C
9	F
10	B

Marker's Use Only	
11	
12	
13	
14	
15	
16 [Bonus]	
Total	

Some advice

Read the whole exam before starting to write, and start by doing the ones you know how to do. The long answer questions are worth much more – spend more time on those.

1. You have 3 hours to complete this exam.
2. Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. **Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets.** If caught with such a device or document, the following may occur: you will be asked to leave the exam immediately and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam.

   By signing the attendance sheet you acknowledge that you will comply with these conditions.
3. Questions 1 through 10 are multiple choice. They are worth 1 point each, you do not have to show work, and no part marks will be given. Please record your answers next to the question number to the right.
4. Questions 11 through 15 require a complete solution, and are worth 6 points each.
5. Question 16 is a bonus question worth 3 points and should only be attempted after all other questions have been completed and checked, since bonus marks are much harder to earn. Spend your time accordingly.
6. **The correct answers for questions 11–16 require justification written legibly and logically: you must convince me that you know why your solution is correct. You must answer these questions in the space provided.** Use the backs of pages if necessary.
7. Where it is possible to check your work, do so.
8. Good luck! Bonne chance!

1. Which of the following are subspaces of $\mathbb{R}^3$?

✓ $U = \{(x-y,\ x+y,\ x-y) \mid x,\ y \in \mathbb{R}\} = \text{span}\{(1,1,1),(-1,1,-1)\}$

✓ $V = \{(x,\ y,\ -y) \mid x,\ y \in \mathbb{R}\} = \text{span}\{(1,0,0),(0,1,-1)\}$

✗ $W = \{(x^2,\ y,\ x+y) \mid x,\ y \in \mathbb{R}\}$ not closed under scalar mult. $(1,0,1) \in W$, $(2,0,2) \notin W$

✓ $X = \{(x,y,z) \mid x-y=0\}$ plane through 0

A. U and V only

B. U and W only

C. W and X only

D. U, W and X only

E. U, V and X only

F. V and W only

2. Let M_{22} denote, as usual, the vector space of real 2×2 matrices, and let A^t denote the transpose of $A \in M_{22}$. The dimension of $V = \{A \in M_{22} \mid A = A^t\}$ is:

A. 0

B. 1

C. 2

D. 3

E. 4

F. V is not a subspace of M_{22}, so we cannot speak of its dimension.

$V = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a,b,d \in \mathbb{R} \right\}$

$= \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ this spanning set is L.I., so $\dim(V) = 3$

3. Complete the following phrase to make a true statement:

"A system of 1100 linear equations in 550 unknowns...".

✗ A. ... always has a solution.

✗ B. ... always has a unique solution.

✓ C. ... may be inconsistent.

✗ D. ... which is consistent always has a unique solution.

✗ E. ... which is consistent never has a unique solution.

✗ F. ... is never consistent.

550

1100

E: $\left[\begin{array}{ccccc|c} 1 & & & & & * \\ & \ddots & & & & * \\ 0 & 0 & 0 & 0 & 1 & * \\ & & & & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ can have unique solution.

4. If $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ then $A^{-1} =$

A. $\begin{bmatrix} 1 & -5 \\ 2 & 3 \end{bmatrix}$

B. $\begin{bmatrix} 1/2 & -1/2 \\ -5 & 2 \end{bmatrix}$

C. $\begin{bmatrix} 1/2 & -5/2 \\ 2 & 3 \end{bmatrix}$

D. $\begin{bmatrix} 0 & -1 \\ 1/2 & 3/2 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix}$

F. $\begin{bmatrix} -1 & -3/2 \\ 5/2 & 2 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 3 & 2 & 1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1/2 & 3/2 \end{array}\right]$$

5. If c_j denotes the j^{th} column of

$$A = \begin{bmatrix} 1 & 4 & -1 & -2 \\ 0 & 0 & 0 & 5 \\ 2 & 8 & -2 & 3 \end{bmatrix}$$

($j = 1, \ldots 4$), which of the following sets is a basis for the column space of A ?

A. $\{c_1\}$

B. $\{c_1, c_2\}$

C. $\{c_1, c_3\}$

D. $\{c_1, c_4\}$

E. $\{c_1, c_2, c_3\}$

F. $\{c_1, c_3, c_4\}$

$$A \sim \begin{bmatrix} 1 & 4 & -1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading 1s are in columns 1 and 4.

6. Suppose A is an $n \times n$ matrix. Among the following statements, which one is **not equivalent** to the others?

A. A is not invertible.

B. $Ax = 0$ has infinitely many solutions.

C. There is $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent.

D. The determinant of A is zero.

E. A is row-equivalent to the identity matrix.

F. The rank of A is less than n.

v_1 v_2 v_3

7. The set of vectors $\{(0,3,4),(1,0,0),(0,4,-3)\}$ is orthogonal. Find the Fourier coefficients (a_1, a_2, a_3) of $v = (0,-1,-1)$, i.e. find real scalars (a_1, a_2, a_3) such that $v = a_1(0,3,4) + a_2(1,0,0) + a_3(0,4,-3)$.

A. $(-\frac{7}{25}, 0, -\frac{1}{25})$
B. $(-\frac{7}{5}, 0, -\frac{1}{5})$
C. $(-7, 0, -1)$
D. $(0,-1,-1)$
E. $(\frac{7}{25}, 0, \frac{1}{25})$
F. $(-\frac{7}{25}, 0, \frac{1}{25})$

$a_1 = \dfrac{v \cdot v_1}{\|v_1\|^2} = \dfrac{-7}{25}$

$a_2 = \dfrac{v \cdot v_2}{\|v_2\|^2} = 0$

$a_3 = \dfrac{v \cdot v_3}{\|v_3\|^2} = \dfrac{-1}{25}$

8. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} = 7$, find $\begin{vmatrix} 3a-5g & 3b-5h & 3c-5j \\ g & h & j \\ d & e & f \end{vmatrix}$.

$= \begin{vmatrix} 3a & 3b & 3c \\ g & h & j \\ d & e & f \end{vmatrix}$

$= 3\begin{vmatrix} a & b & c \\ g & h & j \\ d & e & f \end{vmatrix}$

$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix}$

$= -21$

A. 7
B. 21
C. −21
D. 35
E. −35
F. −7

9. Consider the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ Answer the following questions:

- Is 2 the only eigenvalue of A?
- Is the dimension of $\ker(A - 2I_2)$ equal to 2?
- Is A diagonalizable?

A. Yes, Yes, Yes.
B. Yes, Yes, No.
C. Yes, No, Yes.
D. No, Yes, Yes.
E. No, Yes, No.
F. Yes, No, No.

$C_A(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2$

$\Rightarrow \lambda = 2$ is the only eigenvalue.

$[A - 2I_2 \mid 0] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = s$, $x_2 = 0$

$\Rightarrow \ker(A - 2I_2) = \operatorname{span}\{(1,0)\}$

$\Rightarrow \dim(\ker(A - 2I_2)) = 1$

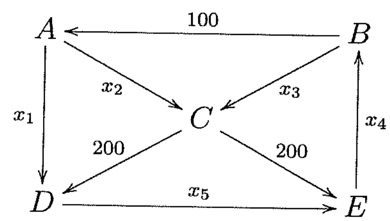
A is not diagonalizable (not enough eigenvectors)

10. Which of the following statements are (always) true?

✗ (1) Each spanning set for $\mathbb{R}^n$ has exactly n vectors. can be larger
✓ (2) If $\{u, v, w\}$ is linearly independent, then $\{u, v\}$ is also linearly independent.
✗ (3) If A is an $n \times n$ matrix, then $\det(-A) = -\det(A)$. only true if n is odd
✗ (4) If A is an $n \times n$ matrix, then $\dim(\operatorname{col}(A)) = n$. could be less
✓ (5) If A is an $n \times n$ matrix, the $\dim(\ker(A)) = n - \operatorname{rank}(A)$. rank-nullity
✓ (6) $\{1, \sin x, \cos x\}$ is linearly independent in $\mathbb{F}(\mathbb{R}) = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$. proved in class

A. All six are true.
B. (2), (5) and (6).
C. (1), (2) and (4).
D. (3), (2) and (6).
E. (4), (5) and (6).
F. (2), (3) and (5).

11. Consider the network of streets and intersections below. The arrows indicate the direction of traffic flow along the <u>one-way streets</u>, and the numbers refer to the <u>exact number</u> of cars observed to enter or leave the intersections during one minute. Each x_i denotes the unknown number of cars which passed along the indicated streets during the same period.



(a) Write down a system of linear <u>equations</u> which describes the the traffic flow, **together with all the constraints** on the variables x_i, $i = 1, \ldots, 5$. (Do not perform any operations on your equations: this is done for you in (b), and *do not simply copy out the equations implicit in (b). You will not get any marks if you do this.*)

In = Out

A: $100 = x_1 + x_2$

B: $x_4 = x_3 + 100$

C: $x_2 + x_3 = 400$

D: $x_1 + 200 = x_5$

E: $x_5 + 200 = x_4$

Constraints: $x_i \geq 0$ for $i = 1, 2, \ldots, 5$ (one-way streets)
$x_i \in \mathbb{Z}$ for $i = 1, \ldots, 5$ (whole numbers of cars).

(Question 11 continued)

(b) The reduced row-echelon form of the augmented matrix from part (a) is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -200 \\ 0 & 1 & 0 & 0 & 1 & 300 \\ 0 & 0 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & -1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Give the general solution. (Ignore the constraints from (a) at this point.)

$x_1 = s - 200$
$x_2 = -s + 300$
$x_3 = s + 100$
$x_4 = s + 200$ $\quad s \in \mathbb{R}$.
$x_5 = s$

(c) Find the maximum and minimum flow along BC, **using your results from (b) and the constraints from (a).**

(You must justify all your answers.)

$x_1 = s - 200 \geq 0 \Rightarrow s \geq 200$
$x_2 = -s + 300 \geq 0 \Rightarrow s \leq 300$
$x_3 = s + 100 \geq 0 \Rightarrow s \geq -100$
$x_4 = s + 200 \geq 0 \Rightarrow s \geq -200$
$x_5 = s \geq 0 \Rightarrow s \geq 0$

For these to all be true simultaneously, we need

$200 \leq s \leq 300$.

$BC = x_3 = s + 100$

$s = 200 \Rightarrow x_3 = 300$ min

$s = 300 \Rightarrow x_3 = 400$ max.

12. Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

(a) Find a basis for col(A), the column space of A.

(b) Find a basis for ker(A).

(c) Give a complete geometric description of ker(A).

(d) Extend your basis of col(A) from (a) to a basis of $\mathbb{R}^4$.

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

a) $\mathrm{col}(A) = \mathrm{span}\{(1,2,1,0), (0,2,2,2)\}$, and this is a basis by the columnspace algorithm.

b) $[A|0] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{aligned} x_1 &= s \\ x_2 &= -3/2\, s \\ x_3 &= s \end{aligned}$

$\Rightarrow \ker(A) = \mathrm{span}\{(1, -3/2, 1)\}$, and this is a basis because it is the basic solution of $[A|0]$.

c) $\ker(A)$ is a line through 0 with direction $(1, -3/2, 1)$.

d) Want v_3, v_4 so that $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ & v_3 & & \\ & v_4 & & \end{bmatrix} \sim I_4$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ & v_3 & & \\ & v_4 & & \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ & v_3 & & \\ & v_4 & & \end{bmatrix}$$

$v_3 = (0,0,1,0)$ works
$v_4 = (0,0,0,1)$

$\Rightarrow \{(1,2,1,0), (0,2,2,2), (0,0,1,0), (0,0,0,1)\}$ is a basis of $\mathbb{R}^4$ which extends the one from (a).

13. Let $U = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + z + w = 0\}$.

(a) Find a basis of U and give the dimension of U.

(b) Find an orthogonal basis of U.

(c) Find the best approximation to $(1, 1, 0, 0)$ by vectors in U.

a) $U = \ker\left(\begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}\right)$

$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \end{array}\right]$ $\quad x_1 = -t-r$, $x_2 = s$, $x_3 = t$, $x_4 = r$

$\Rightarrow U = \operatorname{span}\{\underbrace{(-1,0,1,0)}_{v_1}, \underbrace{(0,1,0,0)}_{v_2}, \underbrace{(-1,0,0,1)}_{v_3}\}$

basic solutions, so this spanning set is a basis.

$\Rightarrow \dim(U) = 3$.

b) $w_1 = v_1 = (-1,0,1,0)$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2} w_1$$

$$= (0,1,0,0) - \frac{0}{2} w_1$$

$$= (0,1,0,0)$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{\|w_1\|^2} w_1 - \frac{v_3 \cdot w_2}{\|w_2\|^2} w_2$$

$$= v_3 - \frac{1}{2} w_1 - 0$$

$$= (-1,0,0,1) - \frac{1}{2}(-1,0,1,0)$$

$$= \left(-\tfrac{1}{2}, 0, -\tfrac{1}{2}, 1\right) \longrightarrow \text{can take } w_3 = (1,0,1,-2).$$

$\Rightarrow \{(-1,0,1,0), (0,1,0,0), (1,0,1,-2)\}$ is an orthogonal basis for U.

(extra page for work)

$$v = (1,1,0,0)$$

c) $$\text{proj}_U(v) = \frac{v \cdot w_1}{\|w_1\|^2} w_1 + \frac{v \cdot w_2}{\|w_2\|^2} w_2 + \frac{v \cdot w_3}{\|w_3\|^2} w_3$$

$$= \frac{-1}{2} w_1 + \frac{1}{1} w_2 + \frac{1}{6} w_3$$

$$= -\frac{1}{2}(-1,0,1,0) + (0,1,0,0) + \frac{1}{6}(1,0,1,-2)$$

$$= \left(\frac{1}{2}+\frac{1}{6},\ 1,\ -\frac{1}{2}+\frac{1}{6},\ \frac{-2}{6}\right)$$

$$= \left(\frac{2}{3},\ 1,\ \frac{-1}{3},\ -\frac{1}{3}\right)$$

14. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

(a) Find the characteristic polynomial $\det(A - \lambda I)$ of A, factor it, and deduce that the eigenvalues of A are 1 and 2.

(b) Find a basis of $E_1 = \{x \in \mathbb{R}^3 \mid Ax = x\}$.

(c) Find a basis of $E_2 = \{x \in \mathbb{R}^3 \mid Ax = 2x\}$.

(d) Find an invertible matrix P such that $P^{-1}AP = D$ is diagonal, and give this diagonal matrix D. Explain why your choice of P is invertible.

a) $C_A(\lambda) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 3\lambda + 2)$

$= (2-\lambda)(\lambda-1)(\lambda-2) \Rightarrow \lambda = 2, \lambda = 1$ are the eigenvalues.

b) $[A - I_3 | 0] = \left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x_1 = -2s \\ x_2 = s \\ x_3 = s \end{array}$

$\Rightarrow E_1 = \text{span}\{(-2,1,1)\}$ basic solutions, so this spanning set is a basis

c) $[A - 2I | 0] = \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\begin{array}{l} x_1 = -t \\ x_2 = s \\ x_3 = t \end{array}$

$\Rightarrow E_2 = \text{span}\{(-1,0,1), (0,1,0)\}$ basic solutions, so this spanning set is a basis

d) $P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

P is invertible because $\dim(E_1) + \dim(E_2) = 3$.

$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

15. State whether each of the following statements is (always) true, or is (possibly) false, in the box after the statement.

- If you say the statement may be false, **you must give an example which shows that it is false.**
- If you say the statement is always true, **you cannot give an example to justify your answer**; you must instead give a clear explanation which works for all possible cases.

(a) If u, v and w are three linearly independent vectors in a vector space E, then $u \in \text{span}\{u - v, v - w\}$.

$u = (1,0,0)$ $\quad u - v = (1,-1,0)$

$v = (0,1,0)$ $\quad v - w = (0,1,-1)$

$w = (0,0,1)$

We solve $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right]$ inconsistent

$\Rightarrow (1,0,0) \notin \text{span}\{(1,-1,0),(0,1,-1)\}$

$\Rightarrow u \notin \text{span}\{u-v, v-w\}$.

ANSWER $\boxed{F}$

(b) Let A be a 3×2 matrix and suppose there is a vector $v \in \mathbb{R}^2$ with $v \neq 0$ and $Av = 0$. Then the columns of A are linearly dependent.

$v = (a, b)$

$A = [v_1 \; v_2]$ where v_1, v_2 are the columns

$0 = Av = [v_1 \; v_2]\begin{bmatrix} a \\ b \end{bmatrix} = av_1 + bv_2$

Since $v \neq 0$, a and b can't both be zero

$\Rightarrow \{v_1, v_2\}$ is L.D.

ANSWER $\boxed{T}$

(c) Let A be a 3×4 matrix. Then every vector in the kernel (or nullspace) of A is orthogonal to every row of A.

Suppose $v \in \ker(A)$, and $A = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$

Then $Av = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} v = \begin{bmatrix} r_1 \cdot v \\ r_2 \cdot v \\ r_3 \cdot v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \; r_1 \cdot v = 0$

$r_2 \cdot v = 0$

$r_3 \cdot v = 0$

ANSWER $\boxed{T}$

(d) If A is an invertible 2×2 matrix, then A is diagonalizable.

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is invertible, but not diagonalizable.

(see Q9)

ANSWER $\boxed{F}$

16. **(3 bonus marks) Make sure you finish and check the rest of the paper before trying this. Bonus marks are much harder to earn.**

In parts (a) and (b), A denotes a symmetric $n \times n$ matrix, i.e., $A = A^t$.

(Note: Your proofs must be valid for *every* symmetric $n \times n$ matrix A, so don't choose n or A !)

(a) Prove that $(Au) \cdot v = u \cdot (Av)$ for all $u, v \in \mathbb{R}^n$, where "$\cdot$" denotes the dot product.

If u, v are written as columns, then $u \cdot v = u^t v$

So $(Au) \cdot v = (Au)^t v = u^t A^t v = u^t A v$

$= u \cdot (Av)$.

(b) Now suppose that $v_\lambda \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue λ, and let

$$W = \{w \in \mathbb{R}^n \mid w \cdot v_\lambda = 0\}.$$

Prove that if $w \in W$, then $Aw \in W$.

Suppose $w \in W$. Then $w \cdot v_\lambda = 0$.

Want to calculate $(Aw) \cdot v_\lambda = w \cdot (Av_\lambda)$ by (a)

$= w \cdot (\lambda v_\lambda)$ because v_λ is an eigenvector

$= \lambda (w \cdot v_\lambda)$

$= \lambda (0)$

$= 0$.

Since $(Aw) \cdot v_\lambda = 0$, we must have $Aw \in W$.