MAT 1302E – Mathematical Methods II

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Review

Final exam organization

1. The seats for you will be determined by the university. There will be lists of students with the seat numbers on the wall. Please take seats according to this list.
2. No books, notes, calculators, cell phones, or other electronic devices are allowed. You have to put all such devices in your bag (NOT in your pocket), the cell phones must be switched off, and have to put your bag in front of the classroom, besides the blackboard.
3. You have to bring your student ID card.
4. Please don't be late! You will be allowed to enter the classroom 10 minutes before the exam starts. You will be allowed to take the exam if you are late, but not more than for 30 minutes. After these 30 minutes, you will have to go to Office of Undergraduate Programs of the Faculty of Science.
5. You cannot hand in your exam before the end of the first hour and during the last 10 minutes (before the actual end of the exam). If there is less than 10 minutes left, please wait until the exam is over.

Final exam organization

1. The exam duration is 3 hours.
2. 2 kinds of questions: short answer (only your final answer is marked) and long answer.
3. There are true/false questions. Unlike at midterms, at the final exam you will receive negative score for false statements incorrectly marked true.
4. Preliminary results will be posted on blackboard.com on January 9 or earlier. After that, there will be two office hours sessions, on January 9 and on January 10, both 13:00–15:00, in the library (101C) of KED. You can see your work and make sure if the marking was correct. Only one person is allowed to enter at a time. After that, your overall course marks will become official.

How to diagonalize matrices

Suppose that $v_1, \ldots, v_n$ are linearly independent eigenvectors of A.

$$P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

$v_1, \ldots, v_n$ are linearly independent, so P is invertible.
$D = P^{-1}AP$
$Pe_i = v_i$
$P^{-1}v_i = e_i$
Each e_i is an eigenvector of D, so D is diagonal. The ith entry on the diagonal is the eigenvalue of v_i.
$A = PDP^{-1}$
$A \neq P^{-1}DP$

How to diagonalize matrices

Summary

If an $n \times n$-matrix A is written in the form

$$A = PDP^{-1},$$

where the columns of P are lineraly independent eigenvectors of A, then D is a diagonal matrix, and the entries on the diagonal are the eigenvalues of A corresponding to these eigenvectors.
In fact, the converse is also true: if $A = PDP^{-1}$, where D is a diagonal matrix, then the columns of P are eigenvectors of A, and the entries on the diagonal of D are the corresponding eigenvalues.

How to diagonalize matrices: procedure

Given an $n \times n$-matrix A,

1. Find its eigenvalues: calculate $\det(A - \lambda I_n)$ and solve $\det(A - \lambda I_n) = 0$. Eigenvalues: $\lambda_1, \dots, \lambda_k$.
2. For each eigenvalue λ_i, find a basis of the corresponding eigenspace: find the general solution of $(A - \lambda_i I_n)x = 0$ and write it in vector parametric form.
3. If the total amount of basis vectors found in the previous step is less than n, then NO, the matrix is NOT DIAGONLIZABLE.
4. If the total amount of basis vectors found in the previous step is n, then P is the matrix whose columns are the eigenvectors and D is the diagonal matrix with the corresponding eigenvalues on the diagonal. The eigenvalues should be written *in the same order* as eigenvalues.
5. Then $A = PDP^{-1}$

Recall: determinants

Notation

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \text{determinant, a number}$$

$$\det \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \text{determinant, a number}$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \text{a matrix}$$

Recall: determinants

Notation

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \end{vmatrix}$$

$$\det \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} = 2 \det \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \end{bmatrix}$$

$$\text{NO:} \quad \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \neq 2 \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \end{bmatrix}$$

$$\text{NO:} \quad \det \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} \neq \det 2 \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 5 \\ 2 & 3 & 4 \end{bmatrix}$$

Recall: properties of determinants

1. If a matrix contains a zero row or a zero column, its determinant is 0.
2. If the rows of a matrix are linearly dependent, the determinant is 0.
3. If the coulmns of a matrix are linearly dependent, the determinant is 0.
4. If the rows of a matrix is linearly independent, it is invertible and its determinant is non-zero.
5. $\det I_n = 1$, $\det \mathbf{0} = 0$.
6. If A is an $n \times n$-matrix, then $\det(\alpha A) = \alpha^n \det A$.
   NO: $\det(\alpha A) \neq \alpha \det A$
7. If A and B are $n \times n$-matrices, then $\det(AB) = (\det A)(\det B)$
8. If A is an invertible matrix, then $\det A^{-1} = 1/(\det A)$
9. $\det A^T = \det A$
10. In general, $\det(A + B) \neq \det A + \det B$!

Recall: determinants and row/column operations

1. $$A \xrightarrow{R_l+\alpha R_k \to R_l} B$$
   $$\det B = \det A$$
   NO: $\det B \neq \alpha \det A, \det B \neq \alpha + \det A$

2. $$A \xrightarrow{\alpha R_k \to R_k} B, \alpha \neq 0$$
   $$\det B = \alpha \det A$$

3. $$A \xrightarrow{R_k \leftrightarrow R_l} B$$
   $$\det B = -\det A$$

The same is true for column operations.

Recall: determinants and row/column operations

$$A \xrightarrow{\alpha R_l + \beta R_k \to R_l} C \quad \text{not a row operation}$$

$$A \xrightarrow{\alpha R_l \to R_l} B \xrightarrow{R_l + \beta R_k \to R_l} C$$

$$\det C = \det B = \alpha \det A$$

$$\text{NO: } \det C \neq \det A$$

The same is true for column operations.

Recall: the absolute value of a complex number

$z = a + bi$
$\overline{z} = a - bi$
$|z| = \sqrt{a^2 + b^2} = \sqrt{z\overline{z}}$
$|z|^2 = a^2 + b^2 = z\overline{z}$
$x^2 - y^2 = (x - y)(x + y)$
$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$

$z = -3 - 4i$
$\overline{z} = -3 + 4i$
$|z|^2 = 3^2 + 4^2 = 25$
$|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

Recall: subspaces

Definition

A subset V of $\mathbb{R}^n$ (or of $\mathbb{C}^n$) is called a *subspace* if:

1. $0 \in V$
2. If $v \in V$ and $w \in V$, then $v + w \in V$
3. If $v \in V$ and $\alpha \in \mathbb{R}$ (or $\alpha \in \mathbb{C}$), then $\alpha v \in V$.

Fact

$$V = \left\{ \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1k}x_k \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2k}x_k \\ \vdots \\ a_{n1}x_n + a_{n2}x_2 + \ldots + a_{nk}x_k \end{bmatrix} \middle| \; x_1, \ldots, x_k \in \mathbb{R} \right\} \text{ is a subspace of } \mathbb{R}^n$$

Recall: subspaces

Warning: if there are constraints on x_i (for example, $x_i > 0$, then *usually* this is not a subspace, *try* to find a counterexample (e. g., two vectors v, w from V such that $v + w \notin V$ or $\alpha v \notin V$ for some $\alpha \in \mathbb{R}$))

$$V = \left\{ \begin{bmatrix} x + 2y \\ y \end{bmatrix} \middle| x, y \in \mathbb{R}, y \geq 0 \right\} \text{ is a NOT subspace of } \mathbb{R}^2$$

The second entry can be 1, but not -1, so you can take v for $x = 0, y = 1$ and $\alpha = -1$ to see that the definition fails.

Recall: subspaces

If there are nonlinear expressions in an entry of a vector (xy, x^2, etc), then *usually* such vectors don't form a subspace, *try* to find a counterexample.

$$\left\{ \begin{bmatrix} 112x + 3y + 4z \\ z \\ x \\ yx \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\} \text{ is a NOT subspace of } \mathbb{R}^4$$

(explained in the previous lecture)

Warning: Sometimes even vectors with nonlinear expressions in the entries form a subspace, the correct answer is only given by the definition.

$$\left\{ \begin{bmatrix} x \\ yz \end{bmatrix} \middle| x, y, z \in \mathbb{R} \right\} \text{ IS a subspace of } \mathbb{R}^2$$

(both entries can be arbitrary, so this is actually the whole $\mathbb{R}^2$)

Recall: subspaces

If there are nonzero scalar summands in an entry of a vector (a number which is not multiplied by a variable), then *usually* such vectors don't form a subspace, *try* to find a counterexample.

$$\left\{ \begin{bmatrix} x+5 \\ x \end{bmatrix} \middle| \, x \in \mathbb{R} \right\} \text{ is a NOT subspace of } \mathbb{R}^2$$

(it does not contain 0)

But:

$$\left\{ \begin{bmatrix} x+5 \\ y \\ y \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\} \text{ IS a subspace of } \mathbb{R}^3$$

(the first entry is arbitrary, the second and the third entries are the same, but also can be an arbitrary number)

Recall: subspaces

Warning: always check that the vectors under consideration are from $\mathbb{R}^n$ for the appropriate value of n

1. $$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \middle| x \in \mathbb{R} \right\} \text{ is a subspace of } \mathbb{R}^2\text{, but NOT of } \mathbb{R}^3$$

2. If A is a 3×5 matrix (3 rows, 5 columns), $Ax = 0$ is a HSLE consisting of 3 equations in 5 variables. So, $\operatorname{Nul} A$ is a subspace of $\mathbb{R}^5$, but NOT of $\mathbb{R}^3$ (and NOT of $\mathbb{R}^4$). On the other hand, $\operatorname{Col} A$ is a subspace of $\mathbb{R}^3$.

$n \times m$-matrix means "n rows, m columns"
$\operatorname{rk} A + \operatorname{nul} A = m$, the number of columns, NOT rows

Recall: traffic flow

See 2014 final exam, problem 15

(c)

Problem: Suppose that the traffic coming from the south to D is limited to 5 cars per minute due to road work. What is the minimal possible amount of cars coming to C from the west?

Solution: To make all traffic nonnegative, we need $x_3 \geq 40$ and $0 \leq x_6 \leq 10$.

Additionally, $x_6 \leq 5$.

So, the possible values for x_6 are: $0 \leq x_6 \leq 5$.

The possible values for $x_4 = 20 - x_6$ are: $15 \leq x_4 \leq 20$.

Answer: At least 15 cars per minute.

Recall: traffic flow

(d)

Problem: Suppose that the traffic coming from the south to D is limited to 15 cars per minute due to road work. What is the minimal possible amount of cars coming to C from the west?

Solution: To make all traffic nonnegative, we need $x_3 \geq 40$ and $0 \leq x_6 \leq 10$.

Additionally, $x_6 \leq 15$.

So, the possible values for x_6 are still: $0 \leq x_6 \leq 10$.

The possible values for $x_4 = 20 - x_6$ are: $10 \leq x_4 \leq 20$.

Answer: At least 10 cars per minute.

Recall: problems with parameters

See 2010 final exam, problem 17
See 2009 final exam, problem 14

Solution sets of systems of linear equations

To find out how many solutions a system of linear equations has:

1. Write the augmented matrix of the system and bring it to EF.
2. If there is a leader in the last column (corresponding to costant terms, i. e. after the vertical line), then there are no solutions. The system is inconsistent.
3. OTHERWISE: If there is a leader in every other column (every other column is a pivot column), then the system has exactly one solution.
4. OTHERWISE: If there is a column without leader (a non-pivot column), then there are infinitely many solutions.

Recall: EF and sets of solutions

Usage in other problems

1. A SLE $Ax = b$ is *inconsistent* implies: b is not in $\operatorname{Col} A$; b is not a linear combination of columns of A.
2. A SLE $Ax = b$ is *consistent* implies: b is in $\operatorname{Col} A$; b is a linear combination of columns of A.
3. A homogeneous SLE $Ax = 0$ is always consistent. The fact that it has exactly one solution means: the columns of A are linearly independent, for square matrices: A is invertible.
4. The fact that a HSLE $Ax = 0$ has infinitely many solutions means: the columns of A are linearly dependent, for square matrices: A is not invertible.

Recall: EF and sets of solutions

1. If an EF of an $n \times m$-matrix A has *a leader in each row* (i. e. there are no zero rows), then $\operatorname{Col} A = \mathbb{R}^n$; all SLEs $Ax = b$ have *at least* one solution.
2. If an EF of an $n \times m$-matrix A does not have a leader in each row (i. e. there are zero rows), then $\operatorname{Col} A \neq \mathbb{R}^n$; some SLEs $Ax = b$ are *inconsistent*.
3. If an EF of an $n \times m$-matrix A has *a leader in each column* (i. e. all columns are pivot columns), then $\dim \operatorname{Col} A = m$; all SLEs $Ax = b$ have *at most* one solution.
4. If an EF of an $n \times m$-matrix A has *a column without a leader* (i. e. a non-pivot column), then $\dim \operatorname{Col} A < m$; some SLEs $Ax = b$ have infinitely many solutions, the others are *inconsistent*, they cannot have a unique solutuion.
5. If A is an $n \times m$-matrix, where $n < m$ (we have less equations than variables), then the HSLE $Ax = 0$ has infinitely many solutions, and a non-homogeneous SLE $Ax = b$ cannot have a *unique* solution.

Recall: EF and sets of solutions

Usually: to answer a Y/N question like "can this system have infinitely many solutions", it is enough to bring matrix to EF.
To find something numerically (for example, a linear dependency relation, i. e. a particular nontrivial solution of a HSLE), one needs to bring the matrix to RREF.
If in doubt, you can always use RREF instead of EF.

Recall: Solving systems of linear equations

Exam 2010, Problem 15

Recall: Leontief input-output model

Exam 2010, Problem 22

Recall: Matrix inversion

Exam 2014, Problem 19

How to find the inverse

1. Only square matrices can be invertible or singular (non-invertible)
2. Write the superaugmented matrix: $[A|I_n]$ and try to bring it to RREF.
3. If at some point you get a line where all entries before te vertical line are zeros, the matrix is *not invertible*.
4. OTHERWISE, the RREF you get should be: $[I_n|B]$. Then $B = A^{-1}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^{-1}$$

Recall: determinant computation

Exam 2014, Prolem 17

Recall: eigenvalues

Exam 2014, Prolem 18