LINEAR ALGEBRA I

LECTURE NOTES

Last modified: November 15, 2016

(These Lecture Notes replace neither the Text Book nor the Lectures)

Part 5

- Eigenvalues, Eigenvectors and the Characteristic Equation
- Diagonalization
- Complex Numbers
- De Moivre's Theorem
- Roots of a Complex Number
- Complex Eigenvalues and Complex Eigenvectors

Eigenvalues, Eigenvectors and the Characteristic Equation

Definition: Let A be an $n \times n$ matrix. A vector $0 \neq x \in R^n$ is called an **eigenvector** of A if $Ax = \lambda x$ for some $\lambda \in R$. The scalar λ is called an **eigenvalue** of A.

Example: Let $A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

$$Ax = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3x.$$

So, x is an eigenvector of A corresponding to an eigenvalue $\lambda = 3$.

$$Ay = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -10 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

So, y is not an eigenvector of A.

λ is an eigenvalue of $A \iff Ax = \lambda x$ for some non-zero vector x.
$\iff (A - \lambda I)x = 0$ for some non-zero vector x. $\iff \det(A - \lambda I) = 0$.

$\det(A - \lambda I) = 0$ is called the **characteristic equation** of A, the scalars λ satisfying this equation are the eigenvalues of A.

When expanded, $\det(A - \lambda I)$ is a polynomial in λ, which has degree n, and called the **characteristic polynomial** of A.

The set of all solutions of $(A - \lambda I)x = 0$ is called the **eigenspace of** A corresponding to λ, and denoted by E_λ. That is,

$$\boxed{E_\lambda = \text{Nul}(A - \lambda I)}$$

Remark: Eigenspace contains the zero vector. But an eigenvector is never the zero vector. An eigenvalue λ might be zero:

$$\boxed{\lambda = 0 \text{ is an eigenvalue of } A \iff A \text{ is not invertible.}}$$

Example: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda = 0 \iff \lambda = 0, 2.$$

Thus, A is not invertible.

Example: Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix} = (4-\lambda)(-\lambda)(-3-\lambda). \\ \det(A - \lambda I) &= 0 \Longleftrightarrow \lambda_1 = -3, \lambda_2 = 0, \lambda_3 = 4. \end{aligned}$$

Remark: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Example: Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$. Find the eigenvalues and the corresponding eigenvectors of A. What are the eigenspace(s)?

Solution: $A - \lambda I = \begin{bmatrix} 3-\lambda & 4 \\ 1 & -\lambda \end{bmatrix}$. The characteristic polynomial of A is

$$\det(A - \lambda I) = \lambda^2 - 3\lambda - 4.$$

The characteristic equation of A is

$$\lambda^2 - 3\lambda - 4 = 0.$$

The eigenvalues of A:

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \Longleftrightarrow \lambda_1 = 4,\ \lambda_2 = -1.$$

Eigenvectors for $\lambda_1 = 4$:

$$(A - \lambda I)x = 0 \Longleftrightarrow \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow x = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, t \in R.$$

$$v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad E_4 = \text{Span}\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}.$$

A basis for E_4: $\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\}$, $\quad \dim E_4 = 1$.

Eigenvectors for $\lambda_2 = -1$:

$$A - (-1)I = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$(A + I)x = 0 \Longleftrightarrow x = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad E_{-1} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

A basis for E_{-1} : $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$, $\quad \dim E_{-1} = 1$.

Theorem: Let A be an $n \times n$ matrix.
If $v_1, v_2, ..., v_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_r$, then $\{v_1, v_2, ..., v_r\}$ is a linearly independent set.

Example: For $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$ we know that:

$$\lambda_1 = 4 \Longrightarrow v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \Longrightarrow v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

By the above theorem, $\{v_1,\ v_2\}$ is a linearly independent set.

Diagonalization

Example: Let A and B be two $n \times n$ matrices such that

$$B = PAP^{-1}$$

for an invertible matrix P. In this case, we say that A is similar to B.

Exercise: Show that the similar matrices have the same eigenvalues.

Solution: We have $B - \lambda I = PAP^{-1} - \lambda PP^{-1} = P(A - \lambda I)P^{-1}$.

$$\begin{aligned} \det(B - \lambda I) &= \det \left(P(A - \lambda I)P^{-1} \right) \\ &= \det P \cdot \det\ (A - \lambda I) \cdot \det\ P^{-1} \\ &= \det P \cdot \det\ (A - \lambda I) \cdot \frac{1}{\det\ P} \\ &= \det\ (A - \lambda I). \end{aligned}$$

Definition: An $n \times n$ matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix D, i.e, if $A = PDP^{-1}$ for some invertible matrix P and a diagonal matrix D. We note that

- $A = PDP^{-1} \Longleftrightarrow P^{-1}AP = D$.
- An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- An $n \times n$ matrix A is diagonalizable if and only if R^n has a basis consisting of eigenvectors of A.
- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.
- $A = PDP^{-1} \Longleftrightarrow A^n = PD^nP^{-1}$.

Example: Is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ diagonalizable?

Solution: Since A is upper triangular, the eigenvalues of A are $\lambda_1 = \lambda_2 = 0$.

$$A - \lambda_i I = A \ \ (i = 1, 2) \ \text{ and } Ax = 0 \Longleftrightarrow x = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}, t \in R.$$

$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda = 0$.

A is a 2×2 matrix but it does not have 2 linearly independent eigenvectors.
Hence, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}$. We know that $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda_1 = 4$, and $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda_2 = -1$. For

$$P = [v_1\ v_2] = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

We have $A = PDP^{-1}$. (Please verify by showing that $AP = PD$.)

$$A = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$A^{10} = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}^{10} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^{10} & 0 \\ 0 & (-1)^{10} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4^{10} & 4^{10} \\ -1 & 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 4^{11}+1 & 4^{11}-4 \\ 4^{10}-1 & 4^{11}+4 \end{bmatrix} = \begin{bmatrix} 838861 & 838860 \\ 209715 & 209716 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 2 \\ 0 & 0 & -2 \end{bmatrix}$

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: Since A is a triangular matrix, eigenvalues of A are the entries on the main diagonal, i.e, $\lambda_1 = 1$, $\lambda_2 = -4$, and $\lambda_3 = -2$.

Eigenvectors corresponding to $\lambda_1 = 1$:

$$A - \lambda_1 I \;=\; A - I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -5 & 2 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, x_1 \in R.$$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda_1 = 1$.

Eigenvectors corresponding to $\lambda_2 = -4$:

$$A - \lambda_2 I = A - (-4)I = A + 4I = \begin{bmatrix} 5 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 4I)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/5)x_2 \\ x_2 \\ 0 \end{bmatrix} = (1/5)x_2 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}.$$

$v_2 = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda_2 = -4$.

Eigenvectors corresponding to $\lambda_3 = -2$:

$$A - \lambda_3 I = A - (-2)I = A + 2I = \begin{bmatrix} 3 & -1 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2I)X = 0 \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/3)x_3 \\ x_3 \\ x_3 \end{bmatrix} = (1/3)x_3 \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

$v_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$ is an eigenvector for $\lambda_3 = -2$.

Since A is a 3×3 matrix and it has three linearly independent eigenvectors,

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}, \; v_3 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

A is diagonalizable.

Let

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \text{ and}$$

$$P = [v_1 \;\; v_2 \;\; v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix}. \text{ Then,}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1/5 & -2/15 \\ 0 & 1/5 & -1/5 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Exercise: Let $A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & -1 & 0 \\ 1 & 0 & 4 \end{bmatrix}$.

a) Find the eigenvalues and corresponding eigenvectors of A.
b) Is A diagonalizable? If yes, write A as $A = PDP^{-1}$, where D is a diagonal matrix.

Solution:

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 3 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} \\ &= (-1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} \\ &= -(1+\lambda)(\lambda^2-6\lambda+5) \\ &= -(1+\lambda)(\lambda-5)(\lambda-1) = 0 \Longleftrightarrow \lambda = -1,\ 5,\ 1. \end{aligned}$$

$\underline{\lambda_1 = -1:}$

$$A - \lambda I = A + I = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A+I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$E_{-1} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, a basis for E_{-1} is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\dim E_{-1} = 1$.

$\underline{\lambda_2 = 5:}$

$$A - \lambda I = A - 5I = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -6 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A-5I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$E_5 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, a basis for E_5 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\dim E_5 = 1$

$\underline{\lambda_3 = 1:}$

$$A - \lambda I = A - I = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A-I)x=0 \Longleftrightarrow x=\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix}=x_3\begin{bmatrix} -3\\ 0\\ 1\end{bmatrix},\ x_3\in R.$$

$E_1=\text{Span}\left\{\begin{bmatrix} -3\\ 0\\ 1\end{bmatrix}\right\}$, a basis for E_1 is $\left\{\begin{bmatrix} -3\\ 0\\ 1\end{bmatrix}\right\}$, $\dim E_1=1$.

A is a 3×3 matrix and the eigenvectors of A

$$\left\{v_1=\begin{bmatrix} 0\\ 1\\ 0\end{bmatrix}, v_2=\begin{bmatrix} 1\\ 0\\ 1\end{bmatrix}, v_3=\begin{bmatrix} -3\\ 0\\ 1\end{bmatrix}\right\}$$

form a basis for R^3.
Thus A is diagonalizable, and $A=PDP^{-1}$, where

$$P=\begin{bmatrix} v_1 & v_2 & v_3\end{bmatrix}=\begin{bmatrix} 0 & 1 & -3\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix},$$

$$D=\begin{bmatrix} \lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}=\begin{bmatrix} -1 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

a) Find the eigenvalues and corresponding eigenvectors of A.
b) Diagonalizable A, if possible.

Solution:

$$\det(A - \lambda I) = -(\lambda - 5)^2(\lambda - 1) = 0 \Longleftrightarrow \lambda_1 = 5,\ \lambda_2 = 1.$$

$\underline{\lambda_1 = 5}$:

$$A - 5I = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 5I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$\lambda_1 = 5: \qquad v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$E_5 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \dim E_5 = 2.$$

$\underline{\lambda_2 = 1}$:

$$A - I = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x_2 \in R.$$

$$v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad \dim E_1 = 1.$$

Thus $A = PDP^{-1}$, where

$$P = [v_1\ v_2\ v_3] = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Alternatively, we have

$$P_1 = \begin{bmatrix} v_2 & v_1 & v_3 \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \Longrightarrow A = P_1 D P_1^{-1}.$$

$$P' = \begin{bmatrix} v_3 & v_1 & v_2 \end{bmatrix}, D' = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix} \Longrightarrow A = P' D' P'^{-1}.$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$. Is A diagonalizable?

Solution:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 3 \\ 1 & -1-\lambda & 2 \\ -1 & 1 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 1 & -2-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & -1-\lambda \\ -1 & 1 \end{vmatrix} \\ &= (1-\lambda)(\lambda^2 + 3\lambda) - 3\lambda \\ &= -\lambda^2(\lambda+2) = 0 \Longleftrightarrow \lambda_1 = 0, \lambda_2 = -2. \end{aligned}$$

$\underline{\lambda_1 = 0}$:

$$A - \lambda I = A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$Ax = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \; x_3 \in R.$$

$$v_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, \quad E_0 = \text{Span}\left\{ \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

$\underline{\lambda_2 = -2}$:

$$A - (-2)I = A + 2I = \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A+2I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \ x_3 \in R.$$

$$v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad E_{-2} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

A is a 3×3 matrix but it has only two linearly independent eigenvectors:

$$\left\{ v_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\},$$

which is not a basis for R^3. Hence, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$. Diagonalize A, if possible.

Solution:

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 0 & 2 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 2 & 0 & 0 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda) \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & 3-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(3-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} \\ &= (4-\lambda)(3-\lambda)(\lambda^2 - 2\lambda - 3) \\ &= (4-\lambda)(3-\lambda)(\lambda-3)(\lambda+1) = 0 \Longleftrightarrow \lambda = -1, 3, 4. \end{aligned}$$

$\underline{\lambda_1 = -1}$:

$$A + I = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A+I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_{-1} = \text{Span}\{v_1\}, \quad \text{dim}E_{-1} = 1.$$

$\underline{\lambda_2 = 3:}$

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 3I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad E_3 = \text{Span}\{v_2, v_3\}, \quad \text{dim}E_3 = 2.$$

$\underline{\lambda_3 = 4:}$

$$A - 4I \ = \begin{bmatrix} -3 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$(A - 4I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, x_2 \in R.$$

$$v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, E_4 = \text{Span}\{v_4\}, \text{dim}E_4 = 1.$$

A is a 4×4 matrix and

$$\left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

a basis for R^4. So A is diagonalizable.

$$P = [v_1 \;\; v_2 \;\; v_3 \;\; v_4] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix};$$

$$A = PDP^{-1}$$

Complex Numbers

A complex number z is of the form

$$z = a + ib, \text{ where } i^2 = -1, \text{ and } a, b \in R.$$

$a =$Re z is the real part part of z, and $b =$Im z is the imaginary part of z.
z is real $\Longleftrightarrow b = 0$, and z is purely imaginary $\Longleftrightarrow a = 0$.

Let $z = a + ib$ and $w = c + id$. Then,

$$\begin{aligned} z + w &= a + c + i(b + d) \\ z - w &= a - c + i(b - d) \\ z \cdot w &= (a + ib) \cdot (c + id) = ac - bd + i(ad + bc). \\ kz &= ka + i(kb), \ k \in R. \end{aligned}$$

$\overline{z} = a - ib$ is called the complex cojugate of $z = a + ib$.

$$\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{z - w} = \overline{z} - \overline{w}, \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}, \quad \overline{\overline{z}} = z,$$
$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}, \quad z + \overline{z} = 2 \text{ Re } z, \quad z - \overline{z} = 2i \text{ Im } z.$$

Let $z = a + ib$ and $w = c + id \neq 0$. Then

$$\begin{aligned} \frac{z}{w} &= \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} = \frac{ac + bd + i(bc - ad)}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2} = x + iy. \end{aligned}$$

The absolute value (modulus) of $z = a + ib$ is

$$|z| = \sqrt{z\ \bar{z}} = \sqrt{a^2 + b^2}$$

We have the following equalities:

$$|zw| = |z| \cdot |w|, \quad z^{-1} = \frac{1}{z} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}$$

Example: Let $z = 9 - 8i$ and $w = 5 + 2i$. Then find $|z|$, $|w|$, $\left|\frac{z}{w}\right|$.

Write $\frac{z}{w}$ in the form of $a + ib$.

Solution:

$$|z| = \sqrt{9^2 + (-8)^2} = \sqrt{145}$$

$$|w| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} = \frac{\sqrt{145}}{\sqrt{29}} = \frac{\sqrt{5 \cdot 29}}{\sqrt{29}} = \sqrt{5}$$

$$\frac{z}{w} = \frac{9 - 8i}{5 + 2i} = \frac{9 - 8i}{5 + 2i} \cdot \frac{5 - 2i}{5 - 2i} = \frac{(45 - 16) + i(-40 - 18)}{25 + 4} = 1 - 2i.$$

Exercises:

1. Let $z = 3 + 4i$ and $w = 5 - 2i$. Express the followings in the form of $a + ib$.
   $(z - w)^2$, $\frac{z}{w}$, $\frac{\overline{z}}{\overline{w}}$, $\frac{1}{z^2}$, $\frac{w}{2z}$.
2. Find: Re $\frac{1}{2 + i}$, Im $\frac{2 + i}{3 + 4i}$, Im $\frac{2 - i}{3 - 4i}$.
3. Write the followings in the form of $a + ib$.
   $\frac{11 + 2i}{4 + 3i}$, $(3 + 5i)(3 - 5i)$, $\frac{6 + i}{7 + 3i}$,
   $(7 - 3i) - (-2 + 4i)$, $\frac{1}{(3 + 4i)^2}$, $\frac{\sqrt{3} + i}{(1 - i)(\sqrt{3} - i)}$.
4. Solve for z if:
   $iz = 2 - i$, $(4 - 3i)\overline{z} = i$.
5. If $z = 1 - 5i$ and $w = 3 + 4i$, find
   $|z|$, $|w|$, $|z/w|$, $|\overline{z/w}|$, and $|\overline{z}/\overline{w}|$,

Trigonometric Ratios

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	0	1	0
$30°$	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$45°$	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$60°$	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$90°$	$\pi/2$	1	0	$-$

Polar Form of a Complex Number

Let $z = a + ib$. Then

$$\begin{aligned}
\cos\theta &= \frac{a}{|z|} \Longrightarrow a = |z|\cos\theta \\
\sin\theta &= \frac{b}{|z|} \Longrightarrow b = |z|\sin\theta \\
z &= a + ib = |z|\cos\theta + i|z|\sin\theta = |z|(\cos\theta + i\sin\theta) = |z|\text{c}i\text{s}\theta,
\end{aligned}$$

where θ is the angle between the positive real axis and the point z, $-\pi < \theta \leq \pi$.

θ is called the argument of z, and denoted by $\theta = \arg z$. (All angles are measured in radians). The polar form of z is

$$z = |z|(\cos\theta + i\sin\theta).$$

Example: Find the polar form of $z = 1 + i$.

Solution: $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$$\left.\begin{aligned}
\cos\theta = \frac{a}{|z|} = \frac{1}{\sqrt{2}} \\
\sin\theta = \frac{b}{|z|} = \frac{1}{\sqrt{2}}
\end{aligned}\right\} \Longrightarrow \theta = \pi/4 \Longrightarrow z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4).$$

Example: What is the polar form of $z = 3 + i3\sqrt{3}$?

Solution: $|z| = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{36} = 6.$

$$\left.\begin{array}{l} \cos\theta = \dfrac{a}{|z|} = \dfrac{3}{6} = \dfrac{1}{2} \\[2ex] \sin\theta = \dfrac{b}{|z|} = \dfrac{3\sqrt{3}}{6} = \dfrac{\sqrt{3}}{2} \end{array}\right\} \Longrightarrow \theta = \pi/3 \Longrightarrow z = 6(\cos\pi/3 + i\sin\pi/3).$$

Example: What is the polar form of $z = \sqrt{2} - i\sqrt{2}$?

Solution: $|z| = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = 2.$

$$\left.\begin{array}{l} \cos\theta = \dfrac{a}{|z|} = \dfrac{\sqrt{2}}{2} \\[2ex] \sin\theta = \dfrac{b}{|z|} = \dfrac{-\sqrt{2}}{2} \end{array}\right\} \Longrightarrow \theta = -\pi/4 \Longrightarrow z = 2(\cos\pi/4 - i\sin\pi/4).$$

Exercises:

1. Write the polar form of the following complex numbers:

   $z = -4 + 4i, \qquad z = 4i, \qquad z = -7, \qquad z = 1, \qquad z = \dfrac{2+2i}{1-i}.$

2. Represent in the form of $a + ib$:

   $z = 4(\cos\pi/2 + i\sin\pi/2), \qquad z = \sqrt{8}(\cos\pi/4 + i\sin\pi/4),$

   $z = 2cis(-\pi/6), \qquad z = \dfrac{2cis(-3\pi/4)}{2cis(5\pi/6)}.$

Complex Multiplication and Division in Polar Form

Let $z_1 = |z_1|cis\theta_1$ and $z_2 = |z_2|cis\theta_2$.

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot cis(\theta_1 + \theta_2)$$
$$\frac{z_1}{z_2} = \frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2),$$
$$\overline{z_1} = r_1 \text{ cis } (-\theta_1)$$

Example: Let $z = 2cis\frac{3\pi}{8}$ and $w = 5cis\frac{2\pi}{3}$. Then,

$$\overline{z} = 2cis\frac{-3\pi}{8} \text{ and } \overline{w} = 5cis\frac{-2\pi}{3},$$
$$z \cdot w = \left(2cis\frac{3\pi}{8}\right)\left(5cis\frac{2\pi}{3}\right) = 2 \cdot 5cis(\frac{3\pi}{8} + \frac{2\pi}{3}) = 10cis(\frac{25\pi}{24}),$$
$$\frac{z}{w} = \frac{2}{5}cis\left(\frac{3\pi}{8} - \frac{2\pi}{3}\right) = \frac{2}{5}cis\left(\frac{-7\pi}{24}\right).$$

Example: $z = cis\left(\frac{\pi}{2}\right)$ and $w = 2cis\left(\frac{-\pi}{3}\right)$.

Find $z \cdot w$, $\frac{z}{w}$, $\overline{z}$ and $\overline{w}$ and write them in their standard forms.
Solution:

$$\begin{aligned}
z \cdot w &= cis\left(\frac{\pi}{2}\right) \cdot 2cis\left(\frac{-\pi}{3}\right) = 2cis\left(\frac{\pi}{2} - \frac{\pi}{3}\right) \\
&= 2cis\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i. \\
\frac{z}{w} &= \frac{cis\left(\frac{\pi}{2}\right)}{2cis\left(\frac{-\pi}{3}\right)} = \frac{1}{2}cis\left(\frac{\pi}{2} - \frac{-\pi}{3}\right) \\
&= \frac{1}{2}cis\left(\frac{5\pi}{6}\right) = \frac{1}{2}\left(\frac{-\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{-\sqrt{3}}{4} + i\frac{1}{4}. \\
\overline{z} &= cis\left(\frac{-\pi}{2}\right) = -i. \\
\overline{w} &= 2cis\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}.
\end{aligned}$$

De Moivre's Theorem

Let n be a positive integer n. Then

$$z^n = (|z|(\cos\theta + i\sin\theta))^n = |z|^n(\cos n\theta + i\sin n\theta).$$

Example: Write $z = (1+i)^{20}$ in the form of $a + ib$.

Solution: $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

$$\begin{aligned}(1+i)^{20} &= (\sqrt{2})^{20}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right) = 2^{10}(\cos(5\pi) + i\sin(5\pi)) \\ &= 2^{10}(\cos\pi + i\sin\pi) = 2^{10}(-1 + i\cdot 0) = -2^{10} = -1024.\end{aligned}$$

Alternatively, we have

$$(1+i)^2 = 2i \Longrightarrow (1+i)^4 = (2i)^2 = -4.$$

$$(1+i)^{20} = ((1+i)^4)^5 = (-4)^5 = -2^{10}.$$

Exercises: Express the following complex numbers in the form of $a + ib$.

1. $z = (2cis(\pi/3))^6$. Ans: 64.
2. $z = (-1+i)^4$.
3. $z = (1-i)^{10}$. Ans: $-32i$
4. $z = (1-i)^{27}$. Ans: $-2^{13}(1+i)$
5. $z = (1+i)^{12}$. Ans: -64
6. $z = (1-i)^6(\sqrt{3}+i)^3$. Ans: -64
7. $z = (\sqrt{3}-i)^9(2-2i)^5$. Ans: $-65536(1+i)$.

Roots of a Complex Number

$$z^n = \alpha\ cis\theta \Longrightarrow z_k = \sqrt[n]{\alpha}\ cis\left(\frac{\theta + 2k\pi}{n}\right), \text{ where } k = 0, 1, 2, ..., n-1.$$

Example: Find all fourth roots of 1.

Solution:

$$z^4 = 1 = \cos 0 + i \sin 0 \Longrightarrow z_k = \text{cis}\left(\frac{0 + 2k\pi}{4}\right), k = 0, 1, 2, 3.$$

$$z_0 = cis0 = 1, \quad z_1 = cis\frac{\pi}{2} = i, \quad z_2 = cis\pi = -1, \quad z_3 = cis\frac{3\pi}{2} = -i.$$

Example: Find all fourth roots of i.

Solution:

$$z^4 = i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} \Longrightarrow z_k = cis\left(\frac{\frac{\pi}{2} + 2k\pi}{4}\right), k = 0, 1, 2, 3.$$

$$z_0 = cis\left(\frac{\pi}{8}\right), \quad z_1 = cis\left(\frac{5\pi}{8}\right), \quad z_2 = cis\left(\frac{9\pi}{8}\right), \quad z_3 = cis\left(\frac{13\pi}{8}\right).$$

Example: Let $z^3 = -8i$. Find z and write it in the standard form.

Solution: $\alpha = |-8i| = 8$, $\theta = -\pi/2$.

$$z_k = \sqrt[3]{8}\ \text{cis}\left(\frac{-\pi/2 + 2k\pi}{3}\right);\ k = 0, 1, 2.$$

$$\begin{aligned} z_0 &= 2\ \text{cis}\ \left(\frac{-\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\frac{-1}{2}\right) = \sqrt{3} - i. \\ z_1 &= 2\ \text{cis}\ (\pi/2) = 2(0 + i) = 2i. \\ z_2 &= 2\ \text{cis}\ (7\pi/6) = 2\left(\frac{-\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i. \end{aligned}$$

Example: Find the roots of $z^2 + z + 1 = 0$.

Solution:

$$z^2 + z + 1 = \left(z + \frac{1}{2}\right)^2 + \frac{3}{4} = 0 \Longrightarrow \left(z + \frac{1}{2}\right)^2 = \frac{-3}{4}$$

$$\Longrightarrow z + \frac{1}{2} = \pm\frac{\sqrt{3}}{2}i \Longrightarrow z = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Example: Find the roots of $z^2 - 4z + 5 = 0$.

Solution:

$$z^2 - 4z + 5 = (z-2)^2 + 1 = 0 \Longrightarrow (z-2)^2 = -1$$

$$\Longrightarrow z - 2 = \pm i \Longrightarrow z = 2 \pm i.$$

Exercises: Find all complex numbers such that:

1. $z^2 = i$	**6.** $z^3 = -27i$
2. $z^3 = i$	**7.** $z^3 = 64i$
3. $z^3 = -i$	**8.** $z^4 = -1$
4. $z^3 = -1$	**9.** $z^4 = 2(i\sqrt{3} - 1)$
5. $z^3 = 27i$	**10.** $z^6 = -64$

Answers:

1. $z_0 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},\ z_1 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$

2. $z_0 = \frac{\sqrt{3}}{2} + i\frac{1}{2},\ z_1 = \frac{-\sqrt{3}}{2} + i\frac{1}{2},\ z_2 = -i.$

3. $z_0 = i,\ z_1 = -\frac{\sqrt{3}}{2} - i\frac{1}{2},\ z_2 = \frac{\sqrt{3}}{2} - i\frac{1}{2}.$

4. $z_0 = -1,\ z_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2},\ z_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}.$

5. $z_0 = -3i,\ z_1 = -\frac{3\sqrt{3}}{2} + i\frac{3}{2},\ z_2 = \frac{3\sqrt{3}}{2} + i\frac{3}{2}.$

6. $z_0 = 3i,\ z_1 = -\frac{3\sqrt{3}}{2} - i\frac{3}{2},\ z_2 = \frac{3\sqrt{3}}{2} - i\frac{3}{2}.$

7. $z_0 = -4i,\ z_1 = -2\sqrt{3} + 2i,\ z_2 = 2\sqrt{3} + 2i.$

8. $\sqrt{2}\ e^{\pi i/6},\ \sqrt{2}\ e^{4\pi i/6},\ \sqrt{2}\ e^{7\pi i/6},\ \sqrt{2}\ e^{10\pi i/6}.$

Note that $e^{i\theta} = \cos\theta + i\sin\theta$.

Complex Eigenvalues and Complex Eigenvectors

Let $\lambda \in C$. If $Ax = \lambda x$ for some non-zero vector x in C^n, then λ is called a complex eigenvalue, and x is called a complex eigenvector.

We note that

$$\det(A - \lambda I) = 0 \Longleftrightarrow Ax = \lambda x$$

for some non-zero vector x in C^n.

If x is a vector in C^n, then the vector $\overline{x}$, whose entries are the complex conjugates of the entries in x, is called the complex conjugate of x.

Real matrices with complex eigenvalues:

Let A be a real $n \times n$ matrix. Let λ be a complex eigenvalue of A with a corresponding eigenvector x in C^n. Then $\overline{\lambda}$ is also an eigenvalue of A with the corresponding eigenvector $\overline{x}$:

$$Ax = \lambda x \Longleftrightarrow \overline{Ax} = \overline{\lambda x} \Longleftrightarrow \overline{A}\,\overline{x} = \overline{\lambda}\,\overline{x} \Longleftrightarrow A\overline{x} = \overline{\lambda}\,\overline{x}$$

Thus the complex eigenvalues of a real matrix A occur in conjugate pairs.

Example: Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.

a) Find the eigenvalues of A, and a basis for each eigenspace in C^2.
b) Diagonalize A.

Solution:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} \Longrightarrow \det(A - \lambda I) = \lambda^2 - 4\lambda + 5 = 0,$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2} = 2 \pm i.$$

<u>$\lambda_1 = 2 + i$:</u>

$$A - (2+i)I = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}.$$

Thus the solution of $(A - (2+i)I)x = 0$ is

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1+i \\ 1 \end{bmatrix},$$

$$v_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}, \quad E_{2+i} = \text{Span}\left\{ \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\}.$$

$\underline{\lambda_2 = 2 - i}$:

Since $\lambda_2 = \overline{\lambda_1}$, an eigenvector corresponding to λ_2 is

$$v_2 = \overline{v_1} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \text{ and } E_{2-i} = \text{Span}\left\{\begin{bmatrix} -1-i \\ 1 \end{bmatrix}\right\}.$$

b) $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 0 & 3+4i \\ 3-4i & 0 \end{bmatrix}$.
Find the eigenvalues and corresponding eigenvectors of A.

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 3+4i \\ 3-4i & -\lambda \end{vmatrix} = \lambda^2 - 25 = 0 \Longrightarrow \lambda_1 = 5, \quad \lambda_2 = -5.$$

$\underline{\lambda_1 = 5}$:

$$A - 5I = \begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-3-4i}{5} \\ 0 & 0 \end{bmatrix}.$$

$$\begin{cases} (A-5I)x = 0 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \dfrac{x_2}{5}\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}. \\ v_1 = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix} \\ E_5 = \text{Span}\left\{\begin{bmatrix} 3+4i \\ 5 \end{bmatrix}\right\}. \end{cases}$$

$\underline{\lambda_2 = -5}$:

$$A - (-5)I = \begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3+4i}{5} \\ 0 & 0 \end{bmatrix}.$$

$$\begin{cases} (A+5I)x = 0 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{x_2}{5}\begin{bmatrix} -(3+4i) \\ 5 \end{bmatrix}. \\ v_2 = \begin{bmatrix} -3-4i \\ 5 \end{bmatrix} \\ E_2 = \text{Span}\left\{\begin{bmatrix} -3-4i \\ 5 \end{bmatrix}\right\}. \end{cases}$$

Note that the matrix A has

- complex entries,
- real eigenvalues,
- complex eigenvectors v_1 and v_2 such that $v_2 \neq \overline{v_1}$.

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} 3+4i & -3-4i \\ 5 & 5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} i & 0 & 1 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$. If possible, diagonalize A.

Solution: $\lambda = i$ is the only eigenvalue of A.

$$A - iI = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - iI)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$E_i = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

Since A is a 3×3 matrix and has only two linearly independent eigenvectors, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$. If possible, diagonalize A.

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & -4 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -2-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2+4) = 0 \Longrightarrow \lambda = 1, -2i, 2i.$$

$\underline{\lambda_1 = 1}$:

$$A - I = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\underline{\lambda_2 = -2i :}$

$$A + 2iI = \begin{bmatrix} 2+2i & 0 & -4 \\ 0 & 1+2i & 0 \\ 2 & 0 & -2+2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1+2i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A + 2iI)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix} \Longrightarrow v_2 = \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}.$$

$\underline{\lambda_3 = 2i :}$
Since A is a real matrix and $\lambda_3 = \overline{\lambda_2}$, an eigenvector for λ_3 is

$$v_3 = \overline{v_2} = \begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}.$$

$A = PDP^{-1}$, where $P = \begin{bmatrix} 0 & 1-i & 1+i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 2i \end{bmatrix}$.

Exercise: Let $A = \begin{bmatrix} -5 & 6 & 2 \\ -3 & 4 & 1 \\ -5 & 5 & 2 \end{bmatrix}$.
Find the eigenvalues and corresponding eigenvectors of A. Diagonalize A, if possible.

Answer: The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = -i$
with corresponding eigenvectors

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 2-i \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 2+i \end{bmatrix}$.