

MATH 1104

FALL 2016

LECTURE NOTES

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(These notes replace neither the textbook nor the lectures)

Part 4

- Determinants
- Properties of Determinants
- Cramer's Rule
- Adjoint of a Matrix and Inverse via Adjoint
- Determinant as the Area of a Parallelogram

DETERMINANTS

Recall that for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $\det(A) = ad - bc$, and A is invertible $\iff \det(A) \neq 0$, and

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Definition: Let $A = (a_{ij})$ be an $n \times n$ matrix, and let A_{ij} denote the submatrix formed by deleting the i th row and j th column of A . Then,

$$\det(A) = (-1)^{1+1}a_{11} \det(A_{11}) + (-1)^{1+2}a_{12} \det(A_{12}) + \cdots + (-1)^{1+n}a_{1n} \det(A_{1n}).$$

This is called the cofactor expansion along the first row.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$. Find $\det(A)$.

Solution: $A_{11} = \begin{bmatrix} 5 & 6 \\ -8 & 9 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -4 & 6 \\ 7 & 9 \end{bmatrix}$, and $A_{13} = \begin{bmatrix} -4 & 5 \\ 7 & -8 \end{bmatrix}$.

$$\det(A_{11}) = 45 - (-48) = 93,$$

$$\det(A_{12}) = -36 - 42 = -78,$$

$$\det(A_{13}) = 32 - 35 = -3.$$

$$\begin{aligned} \det(A) &= (-1)^{1+1}a_{11} \det(A_{11}) + (-1)^{1+2}a_{12} \det(A_{12}) + (-1)^{1+3}a_{13} \det(A_{13}) \\ &= 1 \cdot 1 \cdot 93 - 2 \cdot (-78) + 3 \cdot (-3) = 240. \end{aligned}$$

We can also use the second row of A to calculate $\det(A)$:

$$A_{21} = \begin{bmatrix} 2 & 3 \\ -8 & 9 \end{bmatrix}, A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}, A_{23} = \begin{bmatrix} 1 & 2 \\ 7 & -8 \end{bmatrix}.$$

$$\det(A_{21}) = 42, \det(A_{22}) = -12, \det(A_{23}) = -22.$$

$$\begin{aligned} \det(A) &= (-1)^{2+1}a_{21} \det(A_{21}) + (-1)^{2+2}a_{22} \det(A_{22}) + (-1)^{2+3}a_{23} \det(A_{23}) \\ &= -(-4)(42) + 5(-12) - 6(-22) = 240. \end{aligned}$$

A formula for $\det(A)$ for a 3×3 matrix A :

Let A be a 3×3 matrix. Then

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{11}a_{23}a_{32} + a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33}), \end{aligned}$$

Remark: Use the first and second columns of A to form the fourth and fifth columns, respectively. Then, $\det A$ is obtained by adding the three downward diagonal products and subtracting the three upward diagonal products.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$, what is $\det A$?

Solution:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -4 & 5 & 6 & -4 & 5 \\ 7 & -8 & 9 & 7 & -8 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot (-4) \cdot (-8) \\ &\quad - \left(7 \cdot 5 \cdot 3 + (-8) \cdot 6 \cdot 1 + 9 \cdot (-4) \cdot 2 \right) \\ &= 45 + 84 + 96 - (105 - 48 - 72) = 225 - (-15) = 240. \end{aligned}$$

Example: If $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 1 & 4 & 1 \\ -1 & 2 & 1 & 0 \\ 1 & 0 & 3 & 1 \end{bmatrix}$, what is $\det(A)$?

Solution: We can use cofactor expansion along the first row.

$$\begin{aligned} \det(A_{11}) &= \begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{vmatrix} \text{ (by 1st column)} \\ &= 1 \cdot \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} - 2 \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + 0 \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = 1 - 2 + 0 = -1. \end{aligned}$$

$$\begin{aligned} \det(A_{12}) &= \begin{vmatrix} 2 & 4 & 1 \\ -1 & 1 & 0 \\ 1 & 3 & 1 \end{vmatrix} \text{ (by 2nd row)} \\ &= -(-1) \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 1 + 1 - 0 = 2. \end{aligned}$$

We do not need to calculate $\det(A_{13})$ since $a_{13} = 0$.

$$\begin{aligned} \det(A_{14}) &= \begin{vmatrix} 2 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \text{ (by 3rd row)} \\ &= \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = -7 + 3 \cdot 5 = 8. \end{aligned}$$

$$\begin{aligned} \det(A) &= a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13}) - a_{14} \det(A_{14}) \\ &= 1 \cdot (-1) - 2 \cdot 2 + 0 \cdot \det(A_{13}) - (-1) \cdot 8 = 3. \end{aligned}$$

Example: If $A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ -5 & -8 & 4 & -3 \end{bmatrix}$, find $\det(A)$.

Solution: We can use cofactor expansion along the first row.

$$\det(A) = 4 \det(A_{11}) = 4 \begin{vmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} = 4(-1) \begin{vmatrix} 3 & 0 \\ 4 & -3 \end{vmatrix} = 36$$

- $\det(A) = 4 \cdot (-1) \cdot 3 \cdot (-3) = 36$.
- A square matrix is called upper triangular if all the entries below its main diagonal are zero.
- A square matrix is called lower triangular if all the entries above its main diagonal are zero.
- A matrix is called triangular if it is either upper triangular or lower triangular.
- The identity matrix is both upper triangular and lower triangular.
- If A is a triangular matrix, then $\det(A)$ is the product of the entries on its main diagonal.

Properties of Determinants

Let A and B be two $n \times n$ square matrices.

(i) $\det(A) = \det(A^T)$.

(ii) $\det(AB) = \det(A) \det(B)$

(iii) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Proof of (iii): A is invertible $\iff I = AA^{-1}$.

$$\det(I) = \det(AA^{-1}) \implies 1 \stackrel{\text{(ii)}}{=} \det(A) \det(A^{-1}) \implies \det(A^{-1}) = \frac{1}{\det(A)}.$$

Warning: In general, $\det(A + B) \neq \det(A) + \det(B)$.

Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and $A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$.

$$\det(A) + \det(B) = 1 + 8 = 9 \text{ and } \det(A + B) = 23.$$

So, $\det(A + B) \neq \det(A) + \det(B)$.

The effect of row operations on determinants

Let A be an $n \times n$ square matrix.

- $A \stackrel{R_m \leftrightarrow R_n}{\sim} B \implies \det(B) = -\det(A)$.
- $A \stackrel{R'_n = kR_n}{\sim} B \implies \det(B) = k \det(A)$.
- $A \stackrel{R'_m = R_m + kR_n}{\sim} B \implies \det(B) = \det(A)$.

Note: Column operations have the same effects on determinants as row operations.

Remark 1: Let A be an $n \times n$ matrix.

- If A has two identical rows (or columns), then $\det A = 0$.
- If A has a zero-row (or zero column), then $\det A = 0$.

Remark 2: Let I_n be the $n \times n$ identity matrix and E be an $n \times n$ elementary matrix.

- $I_n \stackrel{R_p \leftrightarrow R_s}{\sim} E \implies \det(E) = -1$.
- $I_n \stackrel{R'_p = kR_p}{\sim} E \implies \det(E) = k$.
- $I_n \stackrel{R'_p = R_p + kR_s}{\sim} E$, then $\det(E) = 1$.

Examples:

$$\begin{vmatrix} 1 & 3 & 2 & 3 \\ 4 & 4 & 5 & 4 \\ 5 & 1 & -3 & 1 \\ -7 & 2 & 8 & 2 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & 3 & 3 \\ 4 & 5 & 5 & -2 \\ 5 & -3 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5,$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \quad (R'_2 = R_2 + 3R_4).$$

Example: If A is an $n \times n$ matrix, then

$$\boxed{\det(kA) = k^n \det(A)}$$

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 5 & 4 & 5 \\ -2 & 0 & 1 \end{bmatrix}$. Find $\det(A)$.

Solution: $\det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 5 & 4 & 5 \\ -2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -6 & 10 \\ 0 & 4 & -1 \end{vmatrix} = \begin{vmatrix} -6 & 10 \\ 4 & -1 \end{vmatrix} = -34.$

$$\begin{aligned} R'_2 &= R_2 - 5R_1 \\ R'_3 &= R_3 + 2R_1 \end{aligned}$$

Or we can do it as

$$\det(A) = \begin{vmatrix} 1 & 2 & -1 \\ 5 & 4 & 5 \\ -2 & 0 & 1 \end{vmatrix} \stackrel{C'_1=C_1+2C_3}{=} \begin{vmatrix} -1 & 2 & -1 \\ 15 & 4 & 5 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ 15 & 4 \end{vmatrix} = -34.$$

Remark: Let A be an $n \times n$ matrix.

i) A is invertible $\iff \det A \neq 0$

ii) $Ax = 0$ has nontrivial solutions $\iff \det A = 0$

Example: Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{bmatrix}$. Find $\det(A)$.

Solution:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 2 & -1 & 1 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} \stackrel{R'_1=R_1+2R_3}{=} \begin{vmatrix} 7 & 0 & 1 & 3 \\ -1 & 0 & 2 & -2 \\ 3 & -1 & 1 & 1 \\ 2 & 0 & -1 & 2 \end{vmatrix} \stackrel{\text{(using 2nd column)}}{=} \begin{vmatrix} 7 & 1 & 3 \\ -1 & 2 & -2 \\ 2 & -1 & 2 \end{vmatrix} \\ &\quad \begin{aligned} R'_2 &= R_2 - 2R_1 \\ R'_3 &= R_3 + R_1 \end{aligned} \\ &= \begin{vmatrix} 7 & 1 & 3 \\ -15 & 0 & -8 \\ 9 & 0 & 5 \end{vmatrix} = - \begin{vmatrix} -15 & -8 \\ 9 & 5 \end{vmatrix} = 3. \end{aligned}$$

Exercise: Let $A = \begin{bmatrix} 1 & 2 & 4 & 5 & 6 \\ 2 & 4 & 6 & 7 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 3 & 5 & 7 & 1 & 1 \\ 0 & 0 & 4 & 2 & 0 \end{bmatrix}$. Find $\det(A)$.

Solution:

$$\begin{aligned} \det(A) &= -3 \begin{vmatrix} 1 & 2 & 4 & 6 \\ 2 & 4 & 6 & 1 \\ 3 & 5 & 7 & 1 \\ 0 & 0 & 4 & 0 \end{vmatrix} = -3(-1)4 \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{vmatrix} \\ &= 12[(4 + 6 + 60) - (72 + 5 + 4)] = 12(70 - 81) = -132. \end{aligned}$$

Example: Let $A = \begin{bmatrix} a & b & c \\ u & v & w \\ x & y & z \end{bmatrix}$, $\det(A) = -6$. Evaluate the determinants of the following matrices.

$$(i) A^{-1}A^T A \quad (ii) 2A \quad (iii) \begin{bmatrix} a & b & 2a+c \\ u & v & 2u+w \\ x & y & 2x+z \end{bmatrix} \quad (iv) \begin{bmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{bmatrix}$$

Solution:

(i)

$$\begin{aligned} \det(A^{-1}A^T A) &= \det(A^{-1}) \cdot \det(A^T) \cdot \det(A) \\ &= \frac{1}{\det(A)} \cdot \det(A) \cdot \det(A) = \det(A) = -6. \end{aligned}$$

(ii) $\det(2A) = 2^3 \det(A) = 8(-6) = -48$.

(iii)

$$\underbrace{\begin{vmatrix} a & b & 2a+c \\ u & v & 2u+w \\ x & y & 2x+z \end{vmatrix}}_{C'_3=C_3-2C_1} = \begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} = \det(A) = -6.$$

(iv)

$$\begin{aligned} \underbrace{\begin{vmatrix} 3a & 3b & 3c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix}}_{R'_1 = \frac{1}{3}R_1} &= 3 \underbrace{\begin{vmatrix} a & b & c \\ 4x & 4y & 4z \\ -u & -v & -w \end{vmatrix}}_{R'_2 = \frac{1}{4}R_2} = 3 \cdot 4 \underbrace{\begin{vmatrix} a & b & c \\ x & y & z \\ -u & -v & -w \end{vmatrix}}_{R'_3 = -R_3} \\ &= 12 \cdot (-1) \underbrace{\begin{vmatrix} a & b & c \\ x & y & z \\ u & v & w \end{vmatrix}}_{R_2 \leftrightarrow R_3} = (-12)(-1) \begin{vmatrix} a & b & c \\ u & v & w \\ x & y & z \end{vmatrix} \\ &= 12 \cdot (-6) = -72. \end{aligned}$$

Example: If A and B are 5×5 matrices and $\det(-2A^T) = 128 = \det(A^3(B^T)^{-1})$, find $\det A$ and $\det B$.

Solution:

$$\det(-2A^T) = (-2)^5 \det(A^T) = -32 \det A = 128 \implies \det A = -128/32 = -4.$$

$$\begin{aligned} \det(A^3(B^T)^{-1}) &= \det(A^3) \det((B^T)^{-1}) = (\det A)^3 \det(B^{-1})^T = (\det A)^3 \det(B^{-1}) \\ &= (-4)^3 \frac{1}{\det B} = 128 \implies \det B = \frac{(-4)^3}{128} = -\frac{1}{2}. \end{aligned}$$

Exercise: Suppose that

$$\det A = -3, \quad \det B = 2 \quad \text{and} \quad \det C = 5.$$

Compute $\det(A^2BC^{-1}B^T)$.

Exercise: Show that $\begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ has an inverse for any a, b and c .

Exercise: If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -3$, then what is $\begin{vmatrix} a_1 + b_1 & a_1 - b_1 & c_1 \\ a_2 + b_2 & a_2 - b_2 & c_2 \\ a_3 + b_3 & a_3 - b_3 & c_3 \end{vmatrix}$?

Solution:

$$\begin{aligned} \underbrace{\begin{vmatrix} a_1 + b_1 & a_1 - b_1 & c_1 \\ a_2 + b_2 & a_2 - b_2 & c_2 \\ a_3 + b_3 & a_3 - b_3 & c_3 \end{vmatrix}}_{C'_2 = C_2 + C_1} &= \begin{vmatrix} a_1 + b_1 & 2a_1 & c_1 \\ a_2 + b_2 & 2a_2 & c_2 \\ a_3 + b_3 & 2a_3 & c_3 \end{vmatrix} = 2 \underbrace{\begin{vmatrix} a_1 + b_1 & a_1 & c_1 \\ a_2 + b_2 & a_2 & c_2 \\ a_3 + b_3 & a_3 & c_3 \end{vmatrix}}_{C'_1 = C_1 - C_2} \\ &= 2 \underbrace{\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}}_{C_1 \leftrightarrow C_2} = -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-2) \cdot (-3) = 6. \end{aligned}$$

CRAMER'S RULE

Let A be an invertible $n \times n$ matrix, and let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. For any b in R^n , $Ax = b$ has a unique solution, and the entries of x are given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \dots, n,$$

where $A_i(b)$ is the matrix obtained from A by replacing the column i by the vector b .

Example: Let $A = \begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Solve $Ax = b$.

Solution: We have $A_1(b) = \begin{bmatrix} 3 & -2 \\ 5 & 1 \end{bmatrix}$ and $A_2(b) = \begin{bmatrix} 7 & 3 \\ 3 & 5 \end{bmatrix}$.

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{13}{13} = 1,$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{26}{13} = 2$$

So we have

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Example: Use Cramer's rule to solve

$$\begin{aligned}x_1 + 2x_3 &= 6 \\-3x_1 + 4x_2 + 6x_3 &= 30 \\-x_1 - 2x_2 + 3x_3 &= 8.\end{aligned}$$

Solution: $A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$,

$$A_1(b) = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, A_2(b) = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, A_3(b) = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}.$$

$$\det A = \underbrace{\begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}}_{C'_3=C_3-2C_1} = \begin{vmatrix} 1 & 0 & 0 \\ -3 & 4 & 12 \\ -1 & -2 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 12 \\ -2 & 5 \end{vmatrix} = 44$$

$$\det A_1(b) = \underbrace{\begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix}}_{C'_1=C_1-3C_3} = \begin{vmatrix} 0 & 0 & 2 \\ 12 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix} = 2 \begin{vmatrix} 12 & 4 \\ -1 & -2 \end{vmatrix} = -40.$$

Similarly, we find that $\det A_2(b) = 72$ and $\det A_3(b) = 152$. Thus

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{-40}{44} = \frac{-10}{11}.$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{72}{44} = \frac{18}{11}.$$

$$x_3 = \frac{\det A_3(b)}{\det A} = \frac{152}{44} = \frac{38}{11}.$$

Example: Use Cramer's rule to solve for x without solving for y , z , and w .

$$\begin{aligned} -y + z + 3w &= 1 \\ x + 2y - z + w &= 2 \\ 3z + 3w &= 0 \\ y + 8z &= 1 \end{aligned}$$

Solution:

$$Ax = b \implies \begin{bmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad A_1(b) = \begin{bmatrix} 1 & -1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 8 & 0 \end{bmatrix}.$$

$$\det A = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = - \underbrace{\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}}_{R'_3 = R_3 + R_1} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18.$$

$$\begin{aligned} \det A_1(b) &= \underbrace{\begin{vmatrix} 1 & -1 & 1 & 3 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix}}_{C'_3 = C_3 - C_4} = \begin{vmatrix} 1 & -1 & -2 & 3 \\ 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 8 & 0 \end{vmatrix} = -3 \underbrace{\begin{vmatrix} 1 & -1 & -2 \\ 2 & 2 & -2 \\ 1 & 1 & 8 \end{vmatrix}}_{\substack{R'_2 = R_2 - 2R_1 \\ R'_3 = R_3 - R_1}} \\ &= -3 \begin{vmatrix} 1 & -1 & -2 \\ 0 & 4 & 2 \\ 0 & 2 & 10 \end{vmatrix} = -3 \begin{vmatrix} 4 & 2 \\ 2 & 10 \end{vmatrix} = -108. \end{aligned}$$

$$\text{So, } x = \frac{\det A_1(b)}{\det A} = \frac{-108}{-18} = 6.$$

Similarly, we obtain $\det A_2(b) = 30$, $\det A_3(b) = -6$ and $\det A_4(b) = 6$, and thus

$$y = \frac{\det A_2(b)}{\det A} = \frac{-5}{3}, \quad z = \frac{\det A_3(b)}{\det A} = \frac{1}{3}, \quad w = \frac{\det A_4(b)}{\det A} = \frac{-1}{3}.$$

A FORMULA FOR A^{-1}

Let A be an invertible $n \times n$ matrix. Let A_{ij} be the submatrix of A formed by deleting row i and column j . Then

$C_{ij} = (-1)^{i+j} \det A_{ij}$ is called the (i, j) -cofactor of A .

The **adjoint** of A is

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{12} & \cdot & \cdot & \cdot & C_{1n} \\ C_{21} & C_{22} & \cdot & \cdot & \cdot & C_{2n} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ C_{n1} & C_{n2} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdot & \cdot & \cdot & C_{n1} \\ C_{12} & C_{22} & \cdot & \cdot & \cdot & C_{n2} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ C_{1n} & C_{2n} & \cdot & \cdot & \cdot & C_{nn} \end{bmatrix}$$

and the inverse of A is

$$A^{-1} = \frac{1}{\det A} \text{adj}A .$$

Recall that the cofactor expansion about the i th row is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

and the cofactor expansion about the j th column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} .$$

Example: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$. Find $\text{adj}A$ and A^{-1} .

Solution:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 12, \quad C_{12} = (-1)^{1+2} \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6,$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -16, \quad C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ -4 & 0 \end{vmatrix} = 4,$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 2, \quad C_{23} = (-1)^{2+3} \begin{vmatrix} 3 & 2 \\ 2 & -4 \end{vmatrix} = 16,$$

$$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 6 & 3 \end{vmatrix} = 12, \quad C_{32} = (-1)^{3+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -10,$$

$$C_{33} = (-1)^{3+3} \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} = 16.$$

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}.$$

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 3 \cdot 12 + 2 \cdot 6 + (-1) \cdot (-16) = 64.$$

Thus,

$$A^{-1} = \frac{1}{\det A} \text{adj}A = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} 3/16 & 1/16 & 3/16 \\ 3/32 & 1/32 & -5/32 \\ -1/4 & 1/4 & 1/4 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 1 & 9 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 17 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ and $A^{-1} = (b_{ij})$. Find b_{31} .

Solution:

$$b_{31} = \frac{C_{13}}{\det A},$$

$$C_{13} = (-1)^{1+3} \det A_{13} = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 17 & 1 \\ 1 & 2 & 1 \end{vmatrix} = (34 + 1 + 0) - (0 + 4 + 1) = 30,$$

$$\det A = \begin{vmatrix} 1 & 9 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 17 & 1 & 1 \\ 1 & 2 & 0 & 1 \end{vmatrix} \stackrel{R'_3=R_3-R_2}{=} \begin{vmatrix} 1 & 9 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 17 & 1 & 1 \\ 0 & -15 & -1 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 9 & 0 \\ 2 & 1 & 1 \\ 0 & -15 & -1 \end{vmatrix} = -32,$$

$$b_{31} = \frac{30}{-32} = -\frac{15}{16}.$$

Exercise: First calculate the $\text{adj}A$, and then use it to find A^{-1} .

$$\text{(i)} A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{(ii)} A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 9 \\ -1 & 3 & 0 \end{bmatrix} \quad \left(A^{-1} = \begin{bmatrix} 3/5 & 0 & -2/5 \\ 1/5 & 0 & 1/5 \\ -11/5 & 1/9 & 1/5 \end{bmatrix} \right)$$

Determinant as the Area of a Parallelogram

Let $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Consider the 2×2 matrix $A = \begin{bmatrix} u & v \end{bmatrix}$.

The area of the parallelogram determined by the vectors u and v is

$$|\det A| = \left| \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \right|.$$

Example: Find the area of the parallelogram defined by $u = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$.

Solution: $A = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \implies \text{Area} = |\det A| = |-15| = 15.$

Example: Find the area of the parallelogram defined by $u = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

Solution: $A = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} \implies \text{Area} = |\det A| = |4 - 24| = 20.$

Example: Find the area of the parallelogram with the vertices

$$(0, -2), (6, -1), (3, 2), (-3, 1).$$

Solution: Translate the parallelogram to one having the origin as a vertex. So, the new vertices are $(0, 0), (6, 1), (3, 4), (-3, 3)$

We note that the vertex $(0, -2)$ is translated to the origin. So, we have

$$\text{Area} = \left| \det \begin{bmatrix} -3 & 6 \\ 3 & 1 \end{bmatrix} \right| = |-21| = 21.$$

It does not matter which vertex is translated to the origin. If the vertex $(6, -1)$ was translated to the origin, then the new vertices would be $(-6, -1), (0, 0), (-3, 3), (-9, 2)$. So,

$$\text{Area} = \left| \det \begin{bmatrix} -3 & -6 \\ 3 & -1 \end{bmatrix} \right| = 21.$$

Exercise: Find the area of the triangle determined by the points $(2, 1)$, $(3, 5)$, $(7, 4)$.

Solution: Let us translate the triangle to one which has the origin as a vertex by translating the point $(2, 1)$ to the origin. So, the new vertices are $(0, 0)$, $(1, 4)$, $(5, 3)$.

$$\text{Area} = \frac{1}{2} \left| \det \begin{bmatrix} 1 & 5 \\ 4 & 3 \end{bmatrix} \right| = \frac{1}{2} |-17| = \frac{17}{2}.$$

Theorem: Let $T : R^2 \rightarrow R^2$ be a linear transformation defined by a 2×2 matrix A . If S is a parallelogram in R^2 , then

$$\boxed{\text{area of } T(S) = |\det A| \cdot \{\text{area of } S\}}$$

Example: Let S be the parallelogram determined by the vectors

$$u = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } v = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Let T be the linear transformation defined by

$$T(x) = Ax, \text{ where } A = \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}.$$

Find the area of $T(S)$.

Solution:

$$\text{area of } S = |\det[u \ v]| = \left| \det \begin{bmatrix} 3 & 2 \\ 2 & 5 \end{bmatrix} \right| = 11.$$

$$\text{area of } T(S) = |\det A| \cdot \{\text{area of } S\} = 10 \cdot 11 = 110 \text{ unit}^2$$

Alternatively, we have

$$\begin{aligned} \text{area of } T(S) &= \left| \det \begin{bmatrix} T(u) & T(v) \end{bmatrix} \right| = \left| \det \begin{bmatrix} 16 & 18 \\ -7 & -1 \end{bmatrix} \right| \\ &= |-16 + 126| = 110 \text{ unit}^2. \end{aligned}$$