MATH 1104
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LECTURE NOTES
Prepared by Ayşe and Şaban Alaca
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(These notes replace neither the textbook nor the lectures)

Part 3

- Subspaces of R^n
- Column Space and Null Space of a Matrix
- Basis for a Subspace
- Dimension of a Subspace
- The Rank Theorem
- Coordinate Vector

SUBSPACES OF R^n

$$X \in R^n \Longleftrightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ or } X = (x_1,\ x_2,\ ...,\ x_n).$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

$$c\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}, \text{ where } c \in R.$$

Definition: A subset H of R^n is called a subspace if it has the following three properties:
(i) The zero vector is in H,
(ii) For each u and v in H, $u + v$ is also in H,
(iii) For each u in H and each scalar c, cu is also in H.

Example: Span$\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ is a line passing through the origin and the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and it is a subspace of R^2.

Remarks:

- If a line does not pass through the origin, it is not a subspace.
- R^n is a subspace of R^n.
  $\{0\}$ is a subspace of R^n.
- If $v_1, v_2, ..., v_k$ are vectors in R^n, then Span$\{v_1, v_2, ..., v_k\}$ is a subspace of R^n. This is called the subspace spanned by $v_1, v_2, ..., v_k$.

Example: Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Show that H =Span$\{e_1, e_2\}$ is a subspace of R^3.
Solution:

(i) $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow 0 \in H.$

(ii) Let $u = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H$ and $v = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H$. Then

$$u + v = (a + c) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (b + d) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H.$$

(iii) $cu = ca \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + cb \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in H.$

So, by (i), (ii) and (iii), H is a subspace.

Example: Determine which of the following sets are subspaces of R^3.

(i) $H_1 = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \middle| \, a \in R \right\}.$

(ii) $H_2 = \left\{ \begin{bmatrix} a \\ 1 \\ 1 \end{bmatrix} \middle| \, a \in R \right\}.$

(iii) $H_3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, b = a + c \right\}.$

(iv) $H_4 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, b = a + c + 1 \right\}.$

Solution: **(i)** Take $a = 0$. Then, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H_1$.

$$u + v = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \in H_1.$$

$$cu = c\begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ca_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \in H_1.$$

Thus, H_1 is a subspace.

Another way to do (i):

$$u \in H_1 \Longrightarrow u = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow H_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, \text{ which is a subspace.}$$

(ii) $$\begin{bmatrix} b \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b+c \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} a \\ 2 \\ 2 \end{bmatrix} \notin H \Longrightarrow H_2 \text{ is not a subspace.}$$

(iii) $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a+c \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \Longrightarrow H_3 = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}.$$

Thus, H_3 is a subspace.

(iv) $$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ a+c+1 \\ c \end{bmatrix} \Longleftrightarrow \begin{cases} a = 0 \\ 1 = 0 \\ c = 0 \end{cases}, \text{ which is a contradiction.}$$

So, zero vector is not in $H_4 \Longrightarrow H_4$ is not a subspace.

Example: Let $H = \left\{ \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} \;\middle|\; t \in R \right\}$ and $W = \left\{ \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} \;\middle|\; s,\ t \in R \right\}$.

Show that H and W are subspaces of R^3 and R^4, respectively.
Solution:

$$u \in H \Longrightarrow u = \begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix} = t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \Longrightarrow H = \text{Span} \left\{ \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

$$w \in W \Longrightarrow w = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \Longrightarrow W = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

Example: Let $W = \left\{ \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} \;\middle|\; a,\ b \in R \right\}$. Is W a subspace of R^3?

Solution:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} \Longleftrightarrow \begin{array}{l} a = 1 \\ a = 6b \\ a = -2b \end{array},$$

which is not satisfied for any a and b. Hence $0 \notin W$, and so W is not a subspace.

Note: If u and v are two vectors in W, then $u + v$ is not in W either.

COLUMN SPACE AND NULL SPACE OF A MATRIX

Definition: Let A be an $m \times n$ matrix.

- The column space A, written as Col$A$, is the set of all linear combinations of the columns of A.

$$\boxed{\text{Col}A = \{b \in R^m : \ Ax = b \text{ for some } x \text{ in } R^n\}}$$

- The null space of A, written as Nul$A$, is the set of all solutions to the homogenous equation $Ax = 0$.

$$\boxed{\text{Nul}A = \{x \in R^n \ : \ Ax = 0\}}$$

Example: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ -5 \\ 0 \end{bmatrix}$.

a) Is b in the column space of A?
b) Find the column space of A.
c) Is $\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$ in the null space of A?

d) Find the null space of A.

Solution: a) $\left[\begin{array}{ccc|c} 1 & -3 & -4 & 3 \\ -3 & 7 & 6 & -5 \\ -4 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & -3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Since the equation $Ax = b$ consistent, $b \in$Col$A$.

$$Ax = b \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3-5t \\ -2-3t \\ t \end{bmatrix}, \ t \in R.$$

b) Since $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$,

$$\text{Col}A = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ -2 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix}\right\}.$$

Remark: Col$A \neq$Span$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$ since $\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix} \notin$ Span$\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

c) $$A\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} \in \text{Nul}A.$$

d)

$$\left[\begin{array}{rrr|r} 1 & -3 & -4 & 0 \\ -3 & 7 & 6 & 0 \\ -4 & 6 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$Ax = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}, t \in R \Longrightarrow \text{NulA} = \text{Span}\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}.$$

Theorem: Let A be an $m \times n$ matrix. Then,

(i) Col$A$ is a subspace of R^m.
(ii) Nul$A$ is a subspace of R^n.

BASIS FOR A SUBSPACE

Definition: A basis for a subspace H of R^n is a linearly independent set (in H) which spans H.

Examples:

- $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for R^2 (**standard basis**).

- $\left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ are bases for R^2 as well.

- $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$,

  $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ are not bases for R^2.

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for R^3 (**standard basis**).

- $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is another basis for R^3.

- $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is not a basis for R^3.

- $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is not a basis for R^3.

- $\{1, x, x^2, ..., x^n\}$ is a basis for P_n, the set of all polynomials of degree at most n.

- $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

  is a basis for $M_{2\times 2}$, the set of all 2×2 matrices.

Example: $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 7 & 6 \\ -4 & 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

Col$A$ =Span$\left\{ \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \\ 6 \end{bmatrix} \right\}$ is a basis for Col$A$.

Nul$A$ =Span$\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul$A$.

DIMENSION OF A SUBSPACE

Definition: The dimension of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H.

The dimension of the subspace $\{0\}$ is 0.

Example: Let $H = \left\{ \begin{bmatrix} a-2b+7c \\ 3a+b+7c \\ 2a-3b+12c \\ 4a+2b+8c \end{bmatrix} \;\middle|\; a,b,c \in R \right\}$.

Find a basis for H. What is the dimension of H?

Solution:

$$\begin{bmatrix} a-2b+7c \\ 3a+b+7c \\ 2a-3b+12c \\ 4a+2b+8c \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \end{bmatrix} + c\begin{bmatrix} 7 \\ 7 \\ 12 \\ 8 \end{bmatrix}.$$

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 12 \\ 8 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 1 & -2 & 7 \\ 3 & 1 & 7 \\ 2 & -3 & 12 \\ 4 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \text{A basis for } H \text{ is } \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -3 \\ 2 \end{bmatrix} \right\}.$$

$\dim H = 2$.

Example: Find a basis for the subspace spanned by the vectors

$$\begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -6 \\ -8 \\ -1 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & -3 & -1 & 5 \\ -2 & 5 & 0 & -6 \\ -4 & 9 & -2 & -8 \\ 3 & -5 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & -7 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A basis for the subspace spanned by the given vectors is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix} \right\}.$

Remark:

$$\begin{bmatrix} -1 \\ 0 \\ -2 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix},$$

$$\begin{bmatrix} 5 \\ -6 \\ -8 \\ -1 \end{bmatrix} = -7 \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} -3 \\ 5 \\ 9 \\ -5 \end{bmatrix}.$$

Definition: The **rank** of a matrix A, denoted by rank$A$, is the dimension of the column space of A.

The Rank Theorem: If a matrix A has n columns, then

- $\boxed{\text{rank}A+\text{dim Nul}A = n}$ or
- $\boxed{\text{dim Col}A+\text{dim Nul}A = n}$.

Example: You are given that

$$A = \begin{bmatrix} 1 & -3 & 2 & 5 \\ -2 & 6 & 0 & -3 \\ 4 & -12 & -4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

(i) Find bases for Col$A$ and Nul$A$.
(ii) Find dim Col$A$ and dim Nul$A$.
(iii) Verify the Rank Theorem.

Solution: (i) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} \right\}$ is a basis for Col$A$.

$Ax = 0 \Longleftrightarrow Bx = 0$.
x_1, x_3 are basic variables and x_2, x_4 are free variables.

$$x_1 = 3x_2 - 2x_3 - 5x_4 = 3x_2 - 2(-7/4)x_4 - 5x_4 = 3x_2 - (3/2)x_4.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 - (3/2)x_4 \\ x_2 \\ (-7/4)x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ 0 \\ -7/4 \\ 1 \end{bmatrix}.$$

A basis for Nul$A$ is $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ -7/4 \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ -7 \\ 4 \end{bmatrix} \right\}$.

(ii) rank$A$=dim Col$A = 2$, dim Nul$A = 2$.

(iii) rank$A +$ dim Nul$A = 4 = \#$ columns of A.

Example: You are given that

$$A = \begin{bmatrix} 3 & -5 & -1 & 4 & 4 \\ -2 & 4 & 2 & 7 & 8 \\ 5 & -9 & -3 & -3 & -4 \\ -2 & 6 & 6 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 3 & -5 & -1 & 4 & 4 \\ 0 & 2 & 4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(i) Find bases for Col$A$ and Nul$A$.
(ii) Find dim Col$A$ and dim Nul$A$.
(iii) Verify the Rank Theorem.

Solution: (i) A basis for Col$A$ is $\left\{ \begin{bmatrix} 3 \\ -2 \\ 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 4 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ -3 \\ 5 \end{bmatrix} \right\}$.

A basis for Nul$A$**:** Need to solve $Ax = 0$.
x_1, x_2, x_4 are basic variables, x_3, x_5 are free variables.

$$x_4 = -x_5, \quad x_2 = -2x_3 - \frac{3}{2}x_5 \text{ and}$$
$$3x_1 = 5x_2 + x_3 - 4x_4 - 4x_5 = 5(-2x_3 - \frac{3}{2}x_5) + x_3 = -9x_3 - \frac{15}{2}x_5$$
$$\Longrightarrow x_1 = -3x_3 - \frac{5}{2}x_5.$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_3 - \frac{5}{2}x_5 \\ -2x_3 - \frac{3}{2}x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \left(\frac{1}{2}\right) \begin{bmatrix} -5 \\ -3 \\ 0 \\ -2 \\ 2 \end{bmatrix}.$$

Then, $\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 0 \\ -2 \\ 2 \end{bmatrix} \right\}$ is a basis for Nul$A$.

(ii) rank$A$=dim Col$A = 3$, dim Nul$A = 2$.

(iii) rank$A +$ dim Nul$A = 5 = \#$ of columns of A.

Example: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 8 & 9 \end{bmatrix}$.

(i) Find bases for Col$A$ and Nul$A$.
(ii) Find dim Col$A$ and dim Nul$A$.
(iii) Verify the Rank Theorem.

Solution: Since $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, A is invertible.

Hence Nul$A = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ and Col$A = R^3$.

(ii) rank$A$=dim Col$A = 3$, dim Nul$A = 0$.

(iii) rank$A +$ dim Nul$A = 3 = \#$ of columns of A.

Remark: Let A be an $n \times n$ matrix. Then, the following conditions are equivalent.

(i) A is invertible.
(ii) dim Nul$A = 0$.
(iii) rank $A = n$.
(iv) dim Col$A = n$.
(v) Col$A = R^n$.
(vi) Nul$A = \{0\}$.
(vii) The columns of A form a basis of R^n.

COORDINATE VECTOR

Definition: Let H be a subspace, $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ a basis for H, and $x \in H$. Then, we know that

$$x = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

for some scalars $c_1,\ c_2, ..., c_n$. The vector

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the coordinate vector of x relative to basis $\mathcal{B}$.

Example: Let H be a subspace with a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} \right\} \text{ and } x = \begin{bmatrix} -8 \\ -1 \\ -3 \end{bmatrix}.$$

Find $[x]_{\mathcal{B}}$.

Solution: We need to solve the equation

$$c_1 \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} -8 \\ -1 \\ -3 \end{bmatrix}.$$

$$\left[\begin{array}{rr|r} -3 & 7 & -8 \\ 1 & 5 & -1 \\ -4 & -6 & -3 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow [x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}.$$

Remark: $x \in R^3$ but $[x]_{\mathcal{B}} \in R^2$.

Example: Let H be a subspace with a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\} \text{ and } x = \begin{bmatrix} -9 \\ 7 \end{bmatrix}.$$

i) Find $[x]_{\mathcal{B}}$.

ii) Find y if $[y]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

Solution: i) We need to solve the equation

$$c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 1 & -3 & -9 \\ -3 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 6 \\ 0 & 1 & 5 \end{array}\right] \Longrightarrow \begin{cases} c_1 = 6, \\ c_2 = 5. \end{cases} \Longrightarrow [x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}.$$

ii)

$$y = 4 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + 6 \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -14 \\ 18 \end{bmatrix}.$$

Exercise: Let H be a subspace with a basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } x = \begin{bmatrix} 8 \\ 4 \\ 8 \\ 2 \end{bmatrix}.$$

Find $[x]_{\mathcal{B}}$.

Solution:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 8 \\ 2 & 0 & 1 & 4 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 4 \\ 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow [x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}.$$

Exercise: Let $u = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Consider the following bases for R^2.

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \; \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{B}_3 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \; \mathcal{B}_4 = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

Show that

$$[u]_{\mathcal{B}_1} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \quad [u]_{\mathcal{B}_2} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad [u]_{\mathcal{B}_3} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad [u]_{\mathcal{B}_4} = \begin{bmatrix} 10 \\ -16 \end{bmatrix}.$$