MATH 1104
FALL 2016
LECTURE NOTES

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(These notes replace neither the textbook nor the lectures)

**Part 1**

- Systems of Linear Equations
- Elementary Row Operations
- Row Echelon Forms
- Vectors Equations
- The Matrix Equations $Ax = b$
- Solution Sets of Linear Systems
- Linear Dependence and Independence

## SYSTEMS OF LINEAR EQUATIONS

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same set of variables, say $x_1, x_2, ..., x_n$.

**Example:**

$$\begin{aligned} 2x_1 - x_2 - 3x_3 &= -1 \\ -2x_1 + 2x_2 + 5x_3 &= 3. \end{aligned}$$

is a linear system and $(x_1, x_2, x_3) = (1, 0, 1)$ is a solution for this system.

$(x_1, x_2, x_3) = (2, -4, 3)$ is also a solution for this system.
The set of all possible solutions is called the **solution set** of the linear system.

Two linear systems are called **equivalent** if they have the same solution set.

**Example:**

$$\begin{array}{rcr} x + y = 4 & | & 8x + 2y = 26 \\ x - y = 2 & | & 13x + 3y = 42 \end{array}$$

are equivalent systems since the solution set for both of them is $\{(3, 1)\}$.

Which of the following system(s) is/are linear?

$$\begin{aligned} x^2 + y &= 7 \\ x - 5y &= 10 \end{aligned}$$

$$\begin{aligned} 3x + xy &= 6 \\ x - y &= 3 \end{aligned}$$

$$\begin{aligned} \sqrt{8}x + y &= 4 \\ 2x - 3y &= 5 \end{aligned}$$

**Remark:** Geometrically, solution of two linear equations in two variables is the intersection of two lines.

**Example:** Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1 : \quad & x + y = 3 \\ l_2 : \quad & x - y = 1 \end{aligned}$$

**Solution:** $(2, 1)$ is the (unique) intersection point of $l_1$ and $l_2$. So, the system has a unique solution.

**Example:** Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1 : \quad & -x + 2y = 1 \\ l_2 : \quad & x - 2y = -3 \end{aligned}$$

**Solution:** Lines are parallel, no intersection. So, the system has no solution.

**Example:** Give a geometric representation of the following system of equations.

$$\begin{aligned} l_1 : \quad & 2x - y = 1 \\ l_2 : \quad & -4x + 2y = -2 \end{aligned}$$

**Solution:** $l_1$ and $l_2$ coincide. The system has infinitely many solutions.

Each of the points

$$(x, y) = (1, 1),\ (2, 3),\ (3, 5)$$

is a solution for this system.

The general solution: $x = t$, $y = -1 + 2t$, $t \in R$.

The solution set is
$\{(x, y) \mid x = t, y = -1 + 2t,\ t \in R\}$ or $\{(t, -1 + 2t) \mid t \in R\}$ or $\{(x, 2x - 1) \mid x \in R\}$.

A system of linear equations has either

**1)** No solution, or
**2)** (Unique) Exactly one solution, or
**3)** Infinitely many solutions.

A system of linear equations is called **consistent** if it has at least one solution, and **inconsistent** if it has no solution.

## Row Echelon Forms (REF)

A **leading entry** of a row refers to the left most non-zero entry (in a non-zero row).

**Echelon form (or row echelon form):**

**1.** Any rows consisting entirely of zeros are placed at the bottom of the matrix.

**2.** The first non-zero element in each row is positioned to the right of the first non-zero element in the previous row.

**Example:**

$$\textbf{i)}\begin{bmatrix} 2 & 1 & 6 & 4 & -2 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \textbf{ii)}\begin{bmatrix} 4 & 3 & -3 & 5 & 0 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 1 & 8 & -4 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \quad \textbf{iii)}\begin{bmatrix} 2 & 5 & 3 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textbf{iv)}\begin{bmatrix} 0 & 2 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 5 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & -1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 7 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & * \end{bmatrix}$$

**Some terms that we use**

**pivot position:** a position of a leading entry in an echelon form of a matrix.

**pivot:** a non-zero number that is in a pivot position.

**pivot column:** a column that contains a pivot position.

**basic variable:** a variable that corresponds to a pivot column.

**free variable:** a variable that is not a basic variable, i.e, a variable that corresponds to a non-pivot column.

**Note:** In an echelon form, in each column that contains a leading entry of some row, all entries below the leading entry are zero.

## Reduced Row Echelon Forms (RREF)

A matrix $A$ is said to be in reduced row echelon form if it satisfies the following conditions.
- It is in row echelon form.
- The leading entry in each nonzero row is 1.
- Each leading 1 is the only nonzero entry in its column.

**Examples:**

$$\textbf{i)}\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \text{(RREF)}$$

$$\textbf{ii)}\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 1 & 1 & 0 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

( RREF if $a_{13} = 0$, not RREF if $a_{13} \neq 0$).

$$\textbf{iii)}\begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & -1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix} \text{( not RREF)}$$

Note: $*$ can be any number.

$$\textbf{iv)}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{(RREF)}$$

$$\textbf{v)}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{(RREF)}$$

$$\textbf{vi)}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{(not RREF, not REF)}$$

**REF** of a matrix is not unique:

**Example:**

$$\begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{(REF)}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{(REF)}$$

$$\sim \begin{bmatrix} 3 & 3 & 3 & 9 \\ 0 & 2 & 6 & 20 \\ 0 & 0 & 3 & 0 \end{bmatrix} \text{(REF)}$$

**RREF** of a matrix is unique:

$$A = \begin{bmatrix} 3 & 1 & 0 & -11 \\ 2 & 1 & -1 & -4 \\ 1 & 1 & 1 & 3 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\text{(RREF)}} = B$$

The matrix $B$ is the only **RREF** of $A$.

**Elementary Row Operations:**

**i)** Multiply a row by a non-zero scalar: $R_2' = 5R_2$

**ii)** Interchange any two rows: $R_1 \longleftrightarrow R_3$

**iii)** Replace a row by the sum of itself and a multiple of another row: $R_3' = R_3 + 4R_2$

**Example:** Solve the following system.

$$\begin{aligned} x + 2y &= 4 \\ -x + 3y + 3z &= -2 \\ y + z &= 0. \end{aligned}$$

**Solution:** The coefficient matrix of the system is

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Its size is $3 \times 3$ (3 rows and 3 columns).
The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

Its size is $3 \times 4$ (3 rows and 4 columns).
We start with the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right] R_2' = R_2 + R_1 \sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & 1 & 0 \end{array}\right] R_2 \longleftrightarrow R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 5 & 3 & 2 \end{array}\right] R_3' = R_3 - 5R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] \quad (REF).$$

Since each column of the coefficient matrix has a pivot, all the variables are basic. Thus, there is a unique solution. Corresponding linear system is

$$\begin{aligned} x + 2y &= 4 \\ y + z &= 0 \\ -2z &= 2, \end{aligned}$$

which has the solution

$$\begin{aligned} &z = -1, \\ &y = -z = 1, \\ &x = -2y + 4 = -2 + 4 = 2. \end{aligned}$$

The solution is $(x, y, z) = (2, 1, -1)$.

The way that we solve the system here is called **back substitution**.

**Remark:** We can also find the **RREF** of the augmented matrix and use it to find the solution.

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ -1 & 3 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{array}\right] R_3' = -\frac{1}{2}R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] R_2' = R_2 - R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] R_1' = R_1 - 2R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad \text{(RREF)}$$

The solution is $(x, y, z) = (2, 1, -1)$.

**Example:** Determine whether the following linear system has a solution.

$$\begin{aligned} x + 2y + z &= 3 \\ x - y + z &= 1 \\ -2x - 4y - 2z &= 4 \end{aligned}$$

**Solution:** The augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ -2 & -4 & -2 & 4 \end{array}\right] \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 + 2R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & 0 & 10 \end{array}\right]$$

The last row says

$$0 \cdot x + 0 \cdot y + 0 \cdot z = 10,$$

which is imposible. So, the system does not have any solutions.

**Example:** Solve the following system of linear equations:

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \\ -5x - 13y + 9z &= 18 \end{aligned}$$

**Solution:**

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ -2 & -5 & 4 & 5 \\ -5 & -13 & 9 & 18 \end{array}\right] \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 5R_1 \end{array} \sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & -3 & -6 & 33 \end{array}\right] R_3' = R_3 - 3R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1' = R_1 + 2R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -7 & 25 \\ 0 & -1 & -2 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2' = -R_2$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -7 & 25 \\ 0 & 1 & 2 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (RREF).$$

$x$ and $y$ are basic variables and $z$ is free variable.

$$x - 7z = 25 \Longrightarrow x = 25 + 7z,$$
$$y + 2z = -11 \Longrightarrow y = -11 - 2z.$$

The general solution in parametric vector form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 + 7z \\ -11 - 2z \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ -11 \\ 0 \end{bmatrix} + z \begin{bmatrix} 7 \\ -2 \\ 1 \end{bmatrix}, \quad z \in R.$$

The system is consistent, and it has infinitely many solutions.

**Exercise:** Solve the following system:

$$\begin{aligned} 2x + 4y - 6z &= 2 \\ y + 3z &= 5 \\ -3x - 5y + 7z &= -3 \end{aligned}$$

**Solution:**

$$\left[\begin{array}{rrr|r} 2 & 4 & -6 & 2 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{array}\right] R_1{}' = \frac{1}{2}R_1 \sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ -3 & -5 & 7 & -3 \end{array}\right] R_3{}' = R_3 + 3R_1$$

$$\sim \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} R_1{}' = R_1 - 2R_2 \\ R_3{}' = R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -9 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -5 & -5 \end{array}\right] R_3{}' = -\frac{1}{5}R_3$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & -9 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} R_1{}' = R_1 + 9R_3 \\ R_2{}' = R_2 - 3R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

The solution of the system: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$

**Warning:** A zero row in an augmented matrix always does not mean infinitely many solutions:

- $\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 3 \\ 0 & -3 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ : There are no solutions.

- $\left[\begin{array}{rrr|r} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ : There are no solutions.

- $\left[\begin{array}{rrr|r} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$ : There is a unique solution.

- $\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 4 & 5 & 3 \\ 0 & 1 & 3 & 2 & -2 & 4 \\ 0 & 0 & 0 & 4 & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ : There are infinitely many solutions.

**Remark:** We can write the system

$$\begin{aligned} x + 2y - 3z &= 3 \\ -2x - 5y + 4z &= 5 \quad (*) \text{ (linear system)} \\ -5x - 13y + 9z &= 18 \end{aligned}$$

in the form of

$$\begin{bmatrix} 1 & 2 & -3 \\ -2 & -5 & 4 \\ -5 & -13 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} \quad (**) \text{ (matrix equation)}$$

or

$$x \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + y \begin{bmatrix} 2 \\ -5 \\ -13 \end{bmatrix} + z \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 18 \end{bmatrix} (***) \text{ (vector equation).}$$

Solving the linear system (*) is the same as solving the matrix equation (**), or vector equation (***).

**Example:** Find the value of the constant $k$ such that the following system has
**i)** no solution,
**ii)** infinitely many solutions,
**iii)** unique solution.

$$\begin{aligned} x+2y-z &= 1 \\ -2x-3y+2z &= -1 \\ -5x-8y+5z &= k \end{aligned}$$

**Solution:**

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ -2 & -3 & 2 & -1 \\ -5 & -8 & 5 & k \end{array}\right] \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 + 5R_1 \end{array} \sim \left[\begin{array}{rrr|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 5+k \end{array}\right] \begin{array}{l} R_1' = R_1 - 2R_2 \\ R_3' = R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3+k \end{array}\right].$$

**i)** $k \neq -3$. **ii)** $k = -3$.
**iii)** There is no $k$ such that the system has a unique solution.

**Example:** Solve the **homogeneous** system:

$$\begin{aligned} x+3y-2z-w &= 0 \\ -2x-5y+4w &= 0 \\ x+4y-6z+w &= 0 \end{aligned}$$

**Solution:**

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ -2 & -5 & 0 & 4 & 0 \\ 1 & 4 & -6 & 1 & 0 \end{array}\right] \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 - R_1 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 1 & -4 & 2 & 0 \end{array}\right] R_3' = R_3 - R_2$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\begin{aligned} y &= 4z-2w, \\ x &= -3y+2z+w = -3(4z-2w)+2z+w = -10z+7w. \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10z+7w \\ 4z-2w \\ z \\ w \end{bmatrix} = z\begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w\begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix}; \quad z, w \in R.$$

**Example:** Solve the **non-homogeneous** system:

$$\begin{aligned} x + 3y - 2z - w &= 2 \\ -2x - 5y + 4w &= 3 \\ x + 4y - 6z + w &= 9 \end{aligned}$$

**Solution:**

$$\left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ -2 & -5 & 0 & 4 & 3 \\ 1 & 4 & -6 & 1 & 9 \end{array}\right] \begin{array}{l} R_2' = R_2 + 2R_1 \\ R_3' = R_3 - R_1 \end{array} \sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 1 & -4 & 2 & 7 \end{array}\right] R_3' = R_3 - R_2$$

$$\sim \left[\begin{array}{rrrr|r} 1 & 3 & -2 & -1 & 2 \\ 0 & 1 & -4 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$y = 4z - 2w + 7,$$

$$x = -3y + 2z + w + 2 = -3(4z - 2w + 7) + 2z + w + 2 = -10z + 7w - 19.$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -10z + 7w - 19 \\ 4z - 2w + 7 \\ z \\ w \end{bmatrix} = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}; \quad z, w \in R.$$

$$\left\{ \begin{array}{c} \text{general solution} \\ \text{of the corresponding} \\ \text{homogeneous system} \end{array} \right\} = v_h = z \begin{bmatrix} -10 \\ 4 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 7 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

$$\left\{ \begin{array}{c} \text{a particular solution of the} \\ \text{non-homogeneous system} \end{array} \right\} = p = \begin{bmatrix} -19 \\ 7 \\ 0 \\ 0 \end{bmatrix}.$$

The general solution of a non-homogeneous system is $\boxed{X = v_h + p}$.

**Example:** Describe all solutions of the homogeneous equation $Ax = 0$ in parametric vector form, where

$$A = \begin{bmatrix} 1 & 6 & 0 & 8 & -1 & -2 \\ 0 & 0 & 1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Solution:** The augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 6 & 0 & 8 & -1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \quad \begin{array}{l} (x_1,\ x_3,\ x_6 \text{ are basic variables}) \\ (x_2,\ x_4,\ x_5 \text{ are free variables}) \end{array}$$

The general solution (in parametric vector form) is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -6x_2 - 8x_4 + x_5 \\ x_2 \\ 3x_4 - 4x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}; \quad x_2, x_4, x_5 \in R.$$

## Linear Combinations of Vectors

**Definition:** If $v_1, v_2, v_3, ..., v_p$ are vectors in $R^n$, and $c_1, c_2, c_3, ..., c_p$ are scalars, then the vector $x$ defined by

$$x = c_1v_1 + c_2v_2 + c_3v_3 + \cdots + c_pv_p$$

is called a **linear combination** of $v_1, v_2, v_3, ..., v_p$.

**Example:** Determine if the vector $b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$ is a linear combination of the vectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \; v_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}.$$

**Solution:**

$$b \in \text{ Span}\{v_1, v_2, v_3\} \Longleftrightarrow xv_1 + yv_2 + zv_3 = b$$

has a solution.

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent if and only if $b \in \text{Span}\{v_1, v_2, v_3\}$.
The general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - 5z \\ 3 - 4z \\ z \end{bmatrix}, z \in R,$$

which means that for any scalar $z$,

$$(2 - 5z)v_1 + (3 - 4z)v_2 + zv_3 = b.$$

For $z = -1$: $7v_1 + 7v_2 - v_3 = b$.

For $z = 0$: $-2v_1 + 3v_2 + 0v_3 = b \Longrightarrow -2v_1 + 3v_2 = b$.

For $z = 1$: $-3v_1 - v_2 + v_3 = b$.

For $z = 3$: $-8v_1 - 5v_2 + 2v_3 = b$.

The set of all linear combinations of $v_1, v_2, ..., v_p$ is called the **span** of $v_1, v_2, v_3, ..., v_p$ and denoted by Span$\{v_1, v_2, v_3, ..., v_p\}$.

- Scalar multiples of $v_i$ are in Span$\{v_1, v_2, ..., v_p\}$.
- Zero vector is in Span$\{v_1, v_2, ..., v_p\}$.
- Span$\{v_1\} = \{cv_1 | c \in R\}$.
- Span$\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ is the $x$-axis.

**Example:** Let

$$b = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}.$$

For what value(s) of $h$, $b$ is in Span$\{v_1,\ v_2\}$?

**Solution:**

$$\left[\begin{array}{cc|c} v_1 & v_2 & b \end{array}\right] = \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{array}\right] R_3{}' = R_3 + 2R_1$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 3 & 2h-5 \end{array}\right] R_3{}' = R_3 - 3R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h+4 \end{array}\right].$$

The system is consistent if and only if $2h + 4 = 0$. So, $b$ is in Span$\{v_1, v_2\}$ if and only if $h = -2$.

**Note:** If $h = -2$, then we have

$$\begin{bmatrix} -2 \\ -3 \\ -5 \end{bmatrix} = -8 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}.$$

**Example:** Solve the matrix equation

$$AX = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \text{ where } A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -2 \\ -2 & 3 & 3 \end{bmatrix}.$$

**Solution:**

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 4 \\ 3 & 7 & -2 & 1 \\ -2 & 3 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & -16 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -14 \end{array}\right].$$

The solution of the system: $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16 \\ 3 \\ -14 \end{bmatrix}$.

**Theorem (or Fact):** The following statements are equivalent for any $m \times n$ matrix $A$.

- The equation $Ax = b$ has a solution for each $b$ in $R^m$,.
- Each $b \in R^m$ is a linear combination of the columns of $A$.
- $A$ has a pivot position in every row.
- The columns of $A$ span $R^m$.

## Linear Dependence and Independence

**Definition:** A set of vectors $\{v_1, v_2, ..., v_k\}$ in $R^n$ is said to be linearly independent if the vector equation

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

has only the trivial solution, i.e, $c_1 = c_2 = \cdots = c_k = 0$.

The set $\{v_1, v_2, ..., v_k\}$ in $R^n$ is said to be linearly dependent if there exist scalars $c_1, c_2, ..., c_k$, not all zero, such that $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$.

**Example:** Decide if the set of vectors $\left\{ \begin{bmatrix} 12 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

**Solution:** We consider the vector equation

$$c_1 \begin{bmatrix} 12 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Take $c_1 = 1$ and $c_2 = 3$. Then

$$\begin{bmatrix} 12 \\ -3 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $v_1$ and $v_2$ are linearly dependent.

A set of two vectors $\{v_1,\ v_2\}$ is linearly dependent if and only if one is a scalar multiple of the other.

**Example:** Decide if the following vectors are linearly independent in $R^2$.

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$$

**Solution:** Consider the vector equation $xv_1 + yv_2 + zv_3 = 0$.

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix}, \; z \in R.$$

Choose $z = 1$. Then

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The vectors are linearly dependent.

Note that none of the above vectors is a multiple of one of the other vectors.

**Theorem:** If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, \; v_2, ..., v_p\}$ in $R^n$ is linearly dependent if $p > n$.

- In $R^2$, the maximum number of linearly independent vectors is 2.
- In $R^3$, the maximum number of linearly independent vectors is 3.

**Exercise:** Are the following vectors linearly independent in $R^3$?

$$\begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 10 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ -7 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$$

**Example:** Decide if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -3 \\ -5 \\ 6 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 5 \\ -6 \end{bmatrix}$$

**Solution:** Consider the equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$.

$$\left[ \begin{array}{ccc|c} v_1 & v_2 & v_3 & 0 \end{array} \right] = \left[ \begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 3 & -5 & 5 & 0 \\ -2 & 6 & -6 & 0 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right].$$

Since each column is a pivot column, there is a unique solution which is

$$x_1 = 0, \; x_2 = 0, x_3 = 0.$$

So, the given vectors are linearly independent.

**Example:** Decide if the following vectors are linearly independent.

$$v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 4 \\ -7 \\ 3 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ 7 \\ -2 \\ -6 \end{bmatrix} \in R^4.$$

**Solution:** Consider the equation $c_1v_1 + c_2v_2 + c_3v_3 = 0$.

$$\left[\begin{array}{ccc|c} v_1 & v_2 & v_3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 4 & 3 & 0 \\ -1 & -7 & 7 & 0 \\ 1 & 3 & -2 & 0 \\ 0 & 2 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then, $c_1 = c_2 = c_3 = 0$. The vectors are linearly independent.

**Remarks:** We put the given vectors in a matrix $A$ as columns.
Then we find an REF of $A$, say $B$.
• If $B$ has a pivot position in each column, then the vectors are linearly independent.
• If $B$ has a non-pivot column, then the vectors are linearly dependent.

**Examples:**

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 2 & 1 & 4 & 3 \\ 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 5 & 7 \\ 0 & 0 & 3 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

• Columns of $A$ are linearly independent.
• Columns of $A$ do not span $R^3$.

• Columns of $B$ are linearly dependent.
• Columns of $B$ span $R^3$.

• Columns of $C$ are linearly independent.
• Columns of $C$ span $R^3$.

• Columns of $D$ are linearly dependent.
• Columns of $D$ do not span $R^4$.

**Example:** Decide if the following vectors are linearly dependent.
If yes, express one of the vectors as a linear combination of the other vectors.

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 4 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

**Solution:**

$$A = \begin{bmatrix} 1 & 1 & 0 & 4 \\ -1 & 0 & 3 & -1 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is a non-pivot column, the vectors are linearly dependent. Note that $x_1$, $x_2$ and $x_3$ are basic variables, and $x_4$ is a free variable. The general solution of the matrix equation $Ax = b$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4x_4 \\ 0 \\ -x_4 \\ x_4 \end{bmatrix}, \quad x_4 \in R.$$

If we choose $x_4 = 1$, then we have: $-4v_1 + 0v_2 - v_3 + v_4 = 0$. So, we have

$$v_4 = 4v_1 + v_3, \;\; v_1 = -\frac{1}{4}v_3 + \frac{1}{4}v_4, \;\; v_3 = -4v_1 + v_4.$$

The vector $v_2$ cannot be expressed as a linear combination of the vectors $v_1$, $v_3$ and $v_4$.

**Remark:** A set $S$ with two or more vectors is linearly dependent $\Longleftrightarrow$ at least one of the vectors in $S$ is expressible as a linear combination of other vectors in $S$.

**Example:** Find the value(s) of $h$ for which the vectors

$$\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ h \end{bmatrix}$$

are linearly dependent.
**Solution:**

$$\begin{bmatrix} 1 & -2 & -1 \\ 3 & -4 & 1 \\ -3 & 1 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 7+h \end{bmatrix}.$$

For $h = -7$, the vectors are linearly dependent.

If $h \in R$ and $h \neq -7$, then the vectors are linearly independent.

**Remarks:**

- A set $S$ with two or more vectors is linearly independent $\Longleftrightarrow$ no vector in $S$ is expressible as a linear combination of other vectors in $S$.
- Zero vector is linearly dependent.
  Any set of vectors containing zero vector is linearly dependent.
- Any set $\{v_1, v_2, ..., v_k\}$ in $R^n$ is linearly dependent if $k > n$, i.e,
  # of vectors > # of entries in one vector.
- Columns of a matrix $A$ are linearly independent $\Longleftrightarrow$ $AX = 0$ has only the trivial solution.

**Exercise:** Find the value(s) of $h$ for which the vectors

$$\begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ h \\ -8 \end{bmatrix}$$

are linearly dependent.

**Solution:** $\begin{bmatrix} 1 & -3 & 4 \\ -5 & 8 & h \\ -2 & 6 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & -7 & 20+h \\ 0 & 0 & 0 \end{bmatrix}.$

Vectors are linearly dependent for all $h$ in $R$.