Chapter 10 Analysis of Variance (ANOVA)

Objective: to compare the means of 3 or more independent populations

- Comparison of the mean reduction in blood pressure for patients randomized to three different treatment groups: drugs A, B, and C
- Compare the mean production time for 3 different process types.

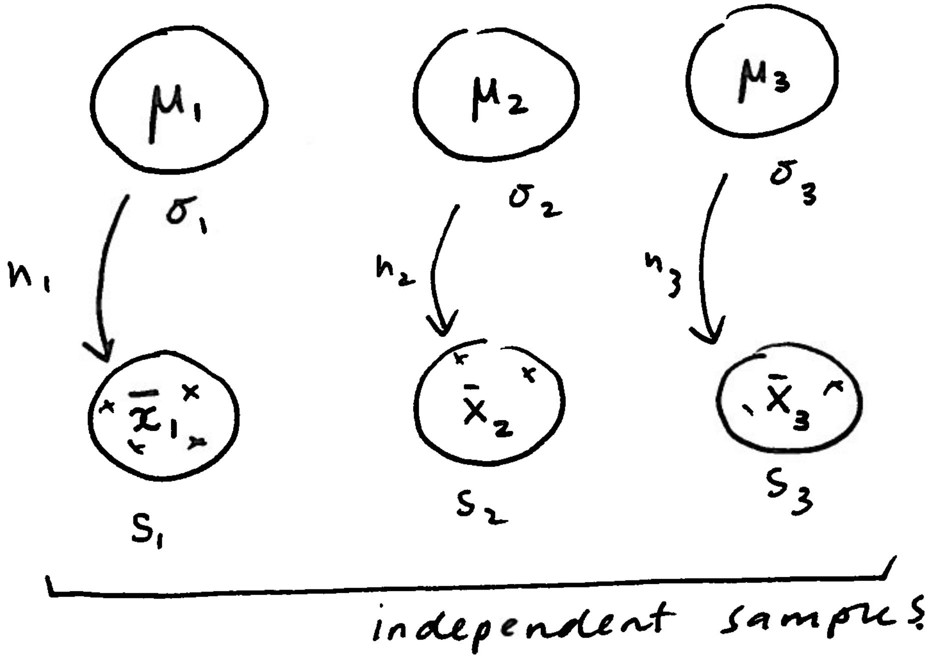
ANOVA is a statistical method that tests the equality of three or more population means by analyzing the variation in the data.

ANOVA Notation:

k = the number of populations under investigation

x_{ij} = measurement on the $\underline{\underline{ith}}$ individual taken from the $\underline{\underline{jth}}$ sample

Population	Population mean	Population SD	Sample size	Sample mean	Sample SD
1	μ_1	σ_1	n_1	$\bar{x}_1$	s_1
2	μ_2	σ_2	n_2	$\bar{x}_2$	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	μ_k	σ_k	n_k	$\bar{x}_k$	s_k
			Total sample size $N = n_1 + n_2 + ... + n_k$	Grand Mean $\bar{\bar{x}} = \frac{1}{N} \sum_{j=1}^{k} \sum_{i=1}^{n_j} x_{ij}$	



X : 10

Idea:

→ point estimate for each pop mean.

→ see how far sample mean are

↳ measure variation between sample means

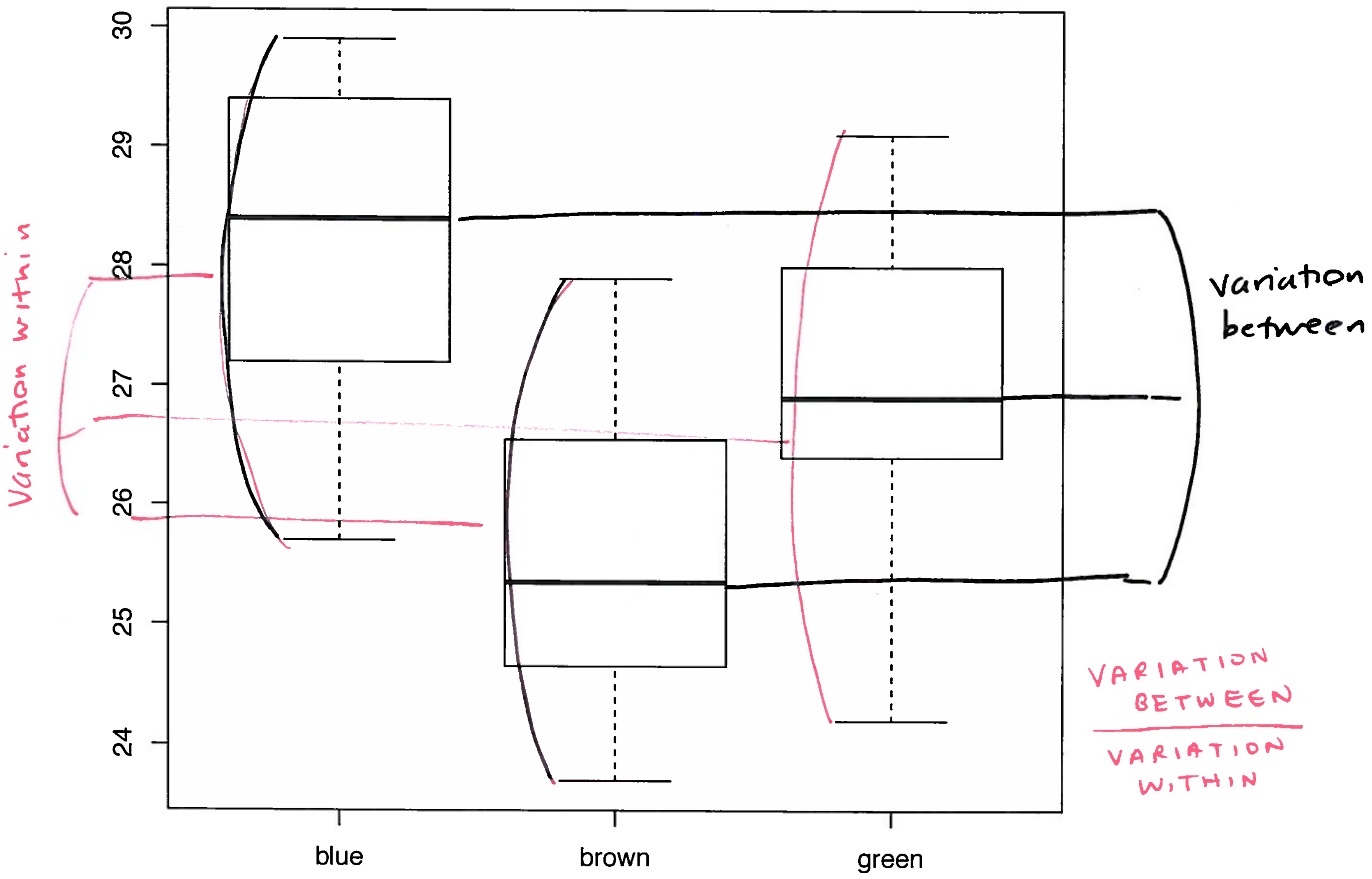
→ variation within each group.

Example 1

A critical flicker frequency (CFF) is the highest frequency at which the flicker in a flickering light source can be detected. There is various flickering light in our environment, such as light from computer screens and fluorescent bulbs. At frequencies above the critical frequency, the light source appears to be continuous even though it is actually flickering. Different people have slightly different flicker “threshold” frequencies. Knowing the critical threshold frequency below which flicker is detected can be important for product manufacturing as well as tests for ocular disease.

Do people with different eye color have different threshold flicker sensitivity? A study ("The effect of iris color on critical flicker frequency," Journal of General Psychology [1973], 91-95) obtained the data from a random sample of 19 subjects. Conduct the appropriate test, set $\alpha = 0.05$

Eye Colour	Threshold Frequency (CCF)	Sample Mean	Sample SD	n
Brown	26.8, 27.9, 23.7, 25.0, 26.3, 24.8, 25.7, 24.5	25.6	1.37	8
Green	26.4, 24.2, 28.0, 26.9, 29.1	26.9	1.84	5
Blue	25.7, 27.2, 29.9, 28.5, 29.4, 28.3	28.2	1.53	6
	Grand Mean =	26.75		



$$\frac{\text{VARIATION BETWEEN}}{\text{VARIATION WITHIN}}$$

Hypotheses:

$$H_0 : \mu_1 = \mu_2 = ... = \mu_k$$
$$H_A : \mu_i \neq \mu_j \text{ for some } i \neq j$$

Assumptions:

1. The k samples drawn from the k populations must be independent of each other.
2. With each sample, the individual observations X_{ij}'s are randomly chosen from population j. (X_{ij}'s are independent)
3. Within each population j, X_{ij}'s follow the Normal distribution with mean μ_j and common variance σ^2. Notice, the k populations have a common variance σ^2. (The best we can do to check this condition is to find the sample standard deviations of our samples and check whether they are close. A common rule of thumb is to see if the ratio between the largest sample standard deviation and the smallest is less than 2.)

The total variation in the data ($SSTotal$ total sum of squares) comes from two sources:

1. variation between groups/ treatments (SST Treatment sum of squares)
2. variation within groups/treatments (SSE Error sum of squares)

$$SSTotal = \overset{\text{between}}{\underline{\underline{SST}}} + \overset{\text{within.}}{\underline{SSE}}$$

$$\underline{\sum_{j=1}^{k}\sum_{i=1}^{n_j}(\underline{\underline{x_{ij}}} - \underline{\underline{\overline{\overline{x}}}})^2} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}(\underset{\uparrow}{\overline{x}_j} - \underset{\uparrow}{\overline{\overline{x}}})^2 + \sum_{j=1}^{k}\sum_{i=1}^{n_j}(\underline{\underline{x_{ij}}} - \underline{\overline{x}_j})^2$$

$$= \sum_{j=1}^{k} n_j(\overline{x}_j - \overline{\overline{x}})^2 + \sum_{j=1}^{k}(n_j - 1)s_j^2$$

$$\boxed{(n_j-1)\,s_j^2 =} \;\Sigma(x_{ij} - \bar{x}_j)^2$$

Last Class:

ANOVA - Analysis of Variance

$X_{ij} \sim N(\mu_j, \sigma^2)$

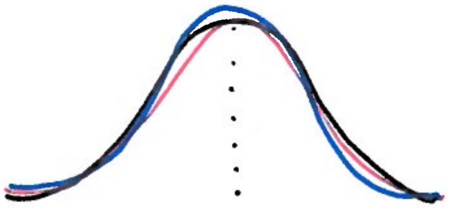
↑ common pop. SD.

Recall:

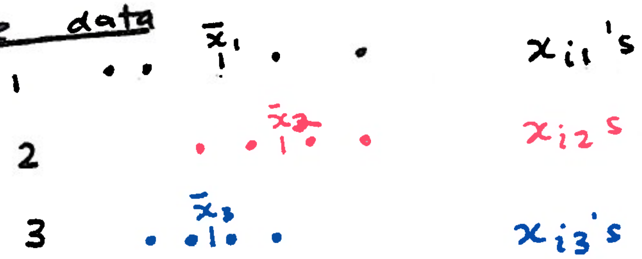
$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n-1} \quad \begin{matrix} \leftarrow SS \\ \leftarrow df \end{matrix}$$

Scenario 1:

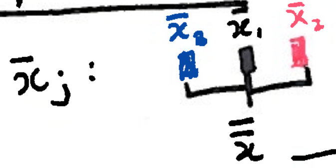
$\mu_1 = \mu_2 = \mu_3$



Sample data

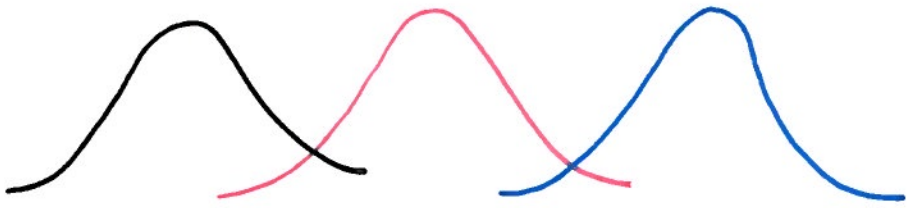


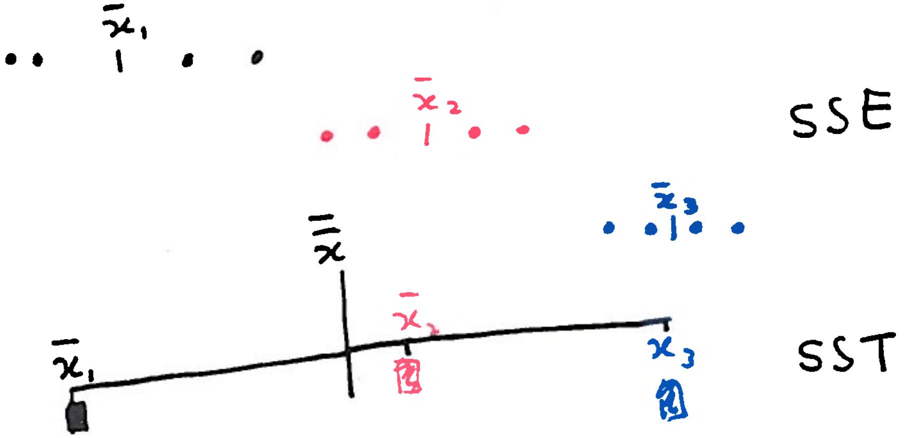
Sample mean



Scenario 2:

Not all means equal





df = # independent obs − # est. parameters.

$$\frac{\text{between}}{\text{within}} \quad \boxed{F_{obs} = \frac{MST}{MSE} = \frac{SST/(K-1)}{SSE/(N-K)}}$$

MSE: pooled variance estimates the σ^2

$$\text{Mean Squares } (MS) = \frac{\text{Sum of Squares}}{\text{degrees of freedom}}$$

1. $MST = \frac{SST}{k-1}$ (Mean Square Treatment)
2. $MSE = \frac{SSE}{N-k}$ (Mean Square Error)

ANOVA test procedure

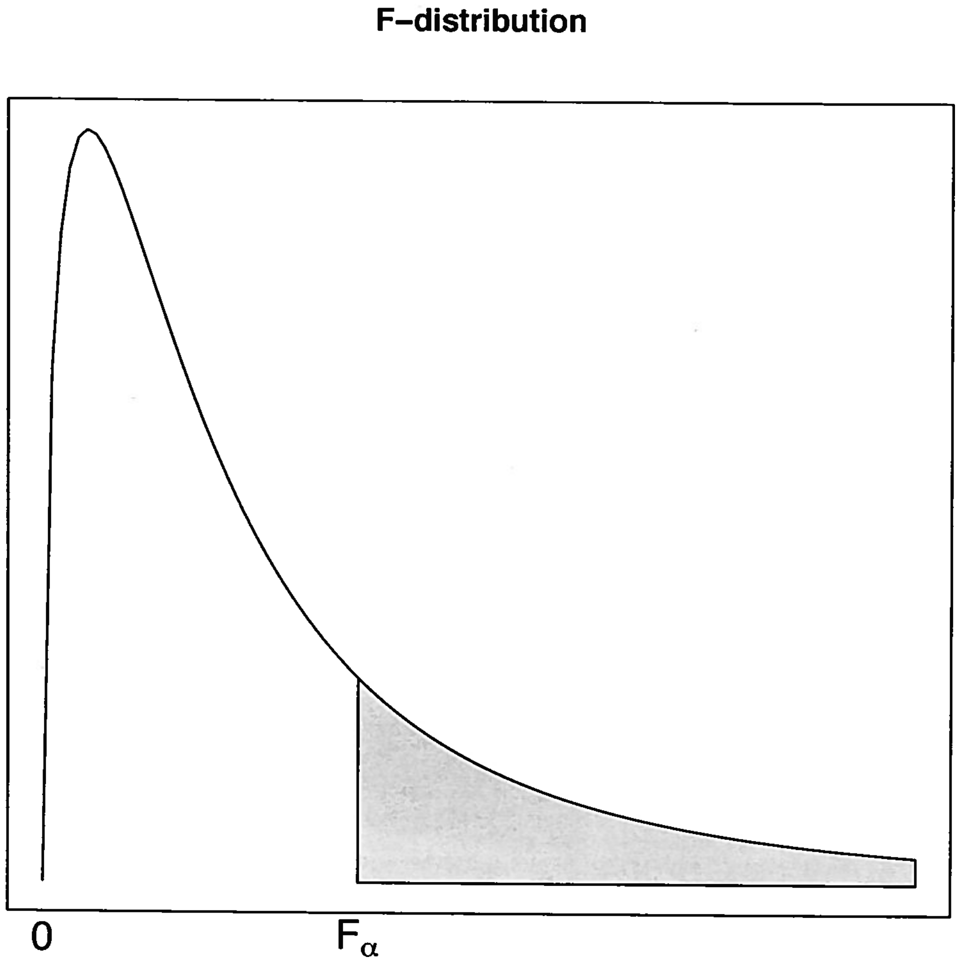
$$H_0 : \mu_1 = \mu_2 = ... = \mu_k$$
$$H_A : \mu_i \neq \mu_j \text{ for some } i \neq j$$

Test statistic:

$$F_{obs} = \frac{MST}{MSE}$$

Under H_0, $F_{obs} = \frac{MST}{MSE}$ follows the F-distribution with

- ○ ν_1 (numerator degrees of freedom $= df(SST) = k - 1$) and
- ○ ν_2 (denominator degrees of freedom $= df(SSE) = N - k$)



Rejection Region:
If $F_{obs} \geq F_\alpha$ we reject H_0
If $F_{obs} < F_\alpha$ we do not reject H_0

When H_0 is rejected, we conclude that at least one pair of population means are significantly different from each other.

The ANOVA table:

Source of Variation	*df*	Sum of Squares	Mean Squares	F-ratio
Treatments	$k-1$	SS_T	$MS_T = \frac{SS_T}{k-1}$	$\frac{MS_T}{MS_E}$
Error	$N-k$	SS_E	$MS_E = \frac{SS_E}{N-k}$	
Total	$N-1$	SS_{Total}		

$H_0: \mu_1 = \mu_2 = \mu_3$ vs. H_A: not all means are equal

Let μ_1 = true mean ~~CCF~~ CFF people with brown eyes
μ_2 = " " " " " green " "
μ_3 = " " blue "

(1) random samples
3 eye colour samples independent

(2) Normality assumption

(3) Equal pop sd: $\frac{1.84}{1.37} < 2$

$$SSE = \sum_{j=1}^{-} \sum_{i=1}^{-} (x_{ij} - \bar{x}_j)^{-} = \sum_{i=1}^{-} (x_{i1} - \bar{x}_1)^{-} + \sum (x_{i2} - x_2) + \sum (x_{i3} - \bar{x}_3)^{-}$$

$$= \cancel{23}.\ 38.31$$

$$SST = \sum_{j=1}^{3} n_j (\bar{x}_j - \bar{\bar{x}})^2 = 8(25.6 - 26.75)^2 + \ldots + 6(28.2 - 26.75)^2$$

$$= 23.00$$

$$SSTotal = SSE + SST = 61.31$$

mSE =

mST =

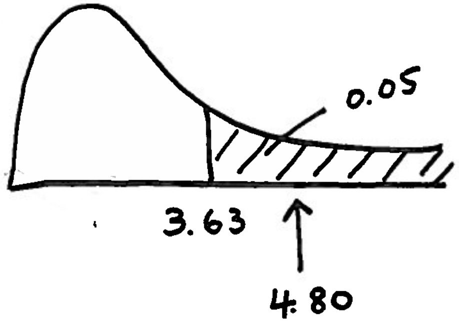
Example 1 cont'd:

		Df	Sum Sq	Mean Sq	F value
K - 1	Colour	3-1=2	23.00	23/2 = 11.5	$\frac{MST}{MSE} = \frac{11.5}{2.38} = 4.80$ ↑
N - K	Error	16	38.31	38.31/16 = 2.38	
N - 1	(Total	18	61.31		

$\alpha = 0.05$

$F_{obs} = 4.80$

$F_{2,16,\alpha=0.05} = 3.63$ (F-table)



$F_{obs} > 3.63$ Reject H_0.

Unlikely to get data like those we observed assuming that CFF is not related to eye colour.

Have enough evidence conclude the true mean CFF in the 3 eye colour pops are not same.