

Chapter 3 Probability

Consider a problem in traffic flow where an engineer wishes to know the number of cyclists crossing a particular intersection within a specified interval of time. This number varies unpredictably from one interval to another reflecting variable cyclist behaviour. For any interval of time, we cannot predict this number with certainty until the time interval has passed. This is an example of a random phenomenon.



Sample Space (denoted S): the set of all possible outcomes of a random phenomenon.

e.g. For tossing a coin, $S = \{H, T\}$

For rolling a die, $S = \{1, 2, 3, 4, 5, 6\}$ $A = \text{roll } 1$

$$P(A) = \frac{1}{6}$$

Event: an outcome or some outcomes from a random phenomenon. Usually denoted with capital letters e.g. A, B, C .

e.g. $A = \{H\}$, $B = \{\text{Roll an even number}\}$

Probability: We use the notation $P(A)$ to denote the probability that an event A will occur.



- ▶ $0 \leq P(A) \leq 1$ (A probability is a number between 0 and 1)
- ▶ $P(A) = 0$ implies that event A is impossible and $P(A) = 1$ implies that event A always occurs

The sum of the probabilities of all the non-overlapping events in a sample space is equal to 1.

① Theoretical (classical)

$$P(A) = \frac{\# \text{ of outcomes in } A}{\text{total } \# \text{ outcomes in } S}$$

if all outcomes are equally likely.

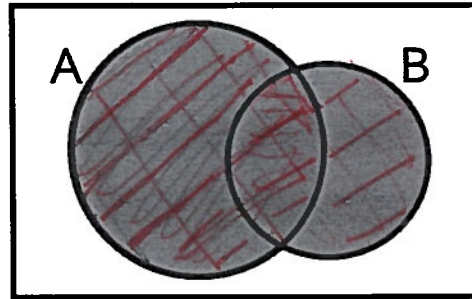
② Empirical (observational)

↳ simulation.

$$P(A) = \frac{\# \text{ times event } A \text{ occurs}}{\text{total } \# \text{ of observed occurrences.}}$$

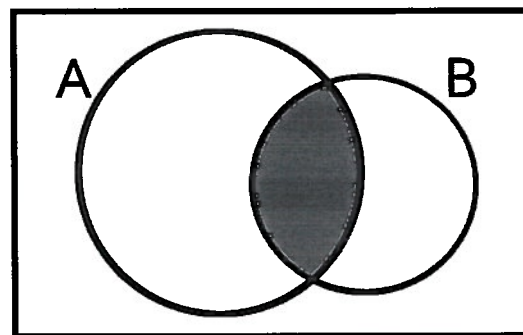
Set Operations

- ▶ The **union** of A and B , $P(A \text{ or } B) = P(A \cup B)$, is the set of all elements belonging to A or B or both \uparrow



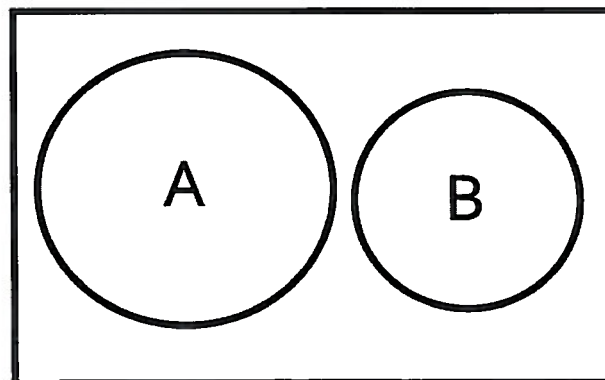
$$P(A) + P(B) - P(A \cap B).$$

- ▶ The **intersection** of A and B , $P(A \text{ and } B) = P(A \cap B)$, is the set of all elements that are common to A and B . \uparrow



Disjoint or Mutually Exclusive Events: when events have no outcomes in common they are said to be *disjoint*. They cannot occur simultaneously i.e. $P(A \text{ and } B \text{ occur simultaneously}) = 0$

$$P(A \cap B) = 0$$



$$P(A \cup B) = P(A) + P(B)$$

$$P(A \cap B) = 0$$

Example 1

Consider an experiment counting the number of left-turning cyclists at an intersection in a group of 100 cyclists. Suppose we define the following events:

Let A be the event that there are 50 or fewer bikes turning left

Let B be the event that between 40 and 60 (inclusive) bikes turn left

Let C be the event that there are 80 or more bikes turning left.

$$S = \{0, 1, 2, \dots, 100\}$$

$$A \cup B = \{0, 1, 2, \dots, 60\}$$

$$A \cap B = \{40, \dots, 50\}$$

$$A = \{0, 1, \dots, 50\}$$

$$B = \{40, 41, \dots, 60\}$$

$$C = \{80, 81, \dots, 100\}$$

A and C are disjoint events.

$$P(A \cap C) = 0.$$

Some properties of probability

1. General Addition Rule:

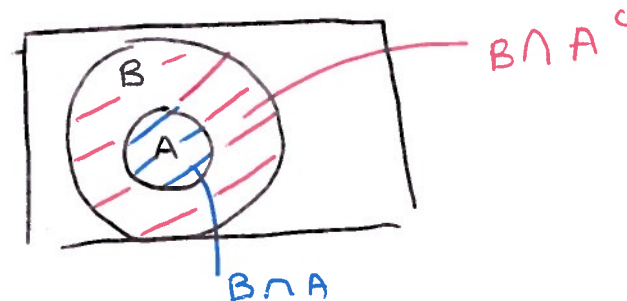
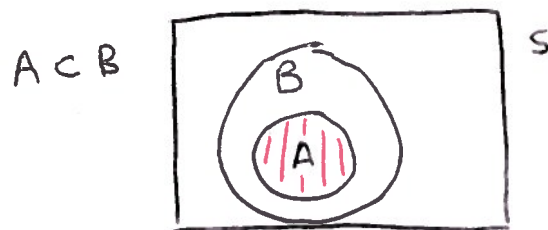
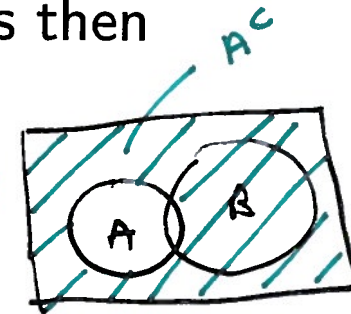
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Notice that if A and B are disjoint events then
 $P(A \cup B) = P(A) + P(B)$

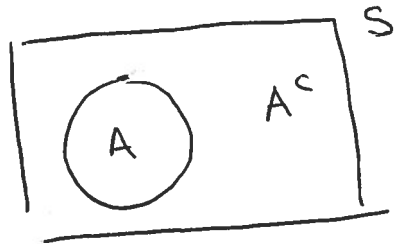
2. Complement Rule: $P(A^c) = 1 - P(A)$ →

3. If $A \subseteq B$ then $P(A \cap B) = P(A)$

4. If $A \subseteq B$ then $P(A) \leq P(B)$



② Complement Rule



$$\boxed{P(A^c) = 1 - P(A)}$$

$$A \cup A^c = S$$

$$\underline{\underline{A \cap A^c = 0}}$$

$$P(S) = 1$$

$$P(A \cup A^c) = P(A) + P(A^c) - \underbrace{P(A \cap A^c)}_0$$

$$= P(A) + P(A^c) = 1$$

$$P(A^c) = 1 - P(A) //$$

④

Since $A \subseteq B$

$$B = (B \cap A) \cup (B \cap A^c)$$

$$= A \cup (B \cap A^c)$$

\approx A and $(B \cap A^c)$ are disjoint

$$P(B) = P(A) + P(B \cap A^c) - 0$$

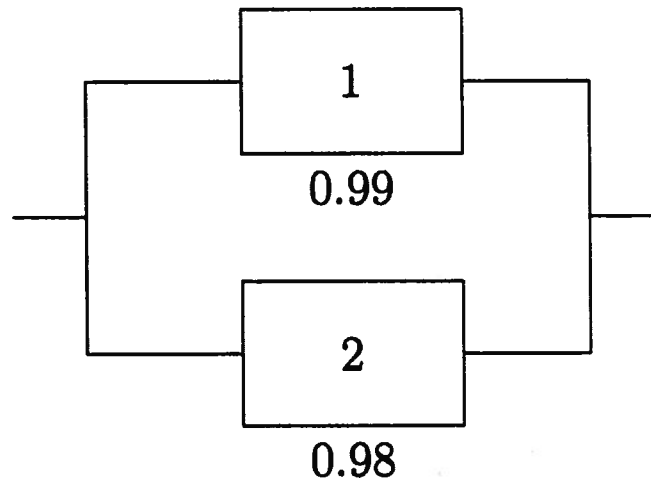
$$\therefore P(B) \geq P(A)$$

$$P(A \cap B) = P(A) \quad \text{③}$$

Example 2

A simplified circuit made up of two components connected in parallel is shown below. The probability that component 1 is operational is 0.99, component 2 is 0.98 and that both components are operational is 0.9702.

Let $A =$ event component 1 works.
 $B =$ " " " 2 works.



- Find the probability that the circuit is not functioning.
- What is the probability that exactly one of the two components is working?

$$a) P(A) = 0.99, P(B) = 0.98, P(A \cap B) = 0.9702$$

$$P(\text{circuit functioning}) = P(A \text{ or } B \text{ or both})$$

$$= P(A \cup B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$$= 0.99 + 0.98 - 0.9702 = \underline{0.9998}$$

$$P(\text{circuit NOT functioning}) = 1 - 0.9998 = 0.0002$$

$$b) P(A \text{ or } B \text{ but NOT both}) = P(A \cup B) - P(A \cap B)$$

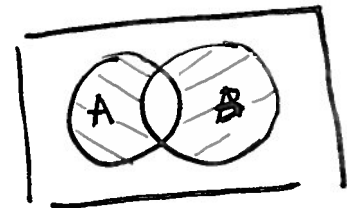
$$= 0.9998 - 0.9702$$

$$= 0.0296$$

Alternate:

$$P(A \cap B^c) + P(A^c \cap B)$$

$$\downarrow \qquad \qquad \downarrow$$
$$P(A) - P(A \cap B) + P(B) - P(A \cap B) = 0.0296$$



Conditional Probability For any two events, A and B with $P(B) > 0$, the **conditional probability** of A given that B has occurred is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

given ↗

Notice, if we rearrange the equation above we get the

Multiplication Rule: $P(A \cap B) = P(B) \times P(A|B)$

Also $P(A \cap B) = P(A) \times P(B|A)$

Suppose equally likely outcomes of experiment.

$$P(B) = \# \text{ outcomes in } B / n$$

$$P(A \cap B) = \# \text{ outcomes in } A \text{ and } B / n$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\# \text{ outcomes in } A \text{ and } B}{\# \text{ of outcomes in } B}.$$

Independence

Two events, A and B are **independent** if knowing that one occurs does not change the probability that the other occurs. That is:

$$\boxed{P(A) = P(A|B) = P(A|B^c)} \quad (1)$$

The probability of A is the same when we are given that B has occurred. Equivalently, A and B are independent if $P(B) = P(B|A) = P(B|A^c)$

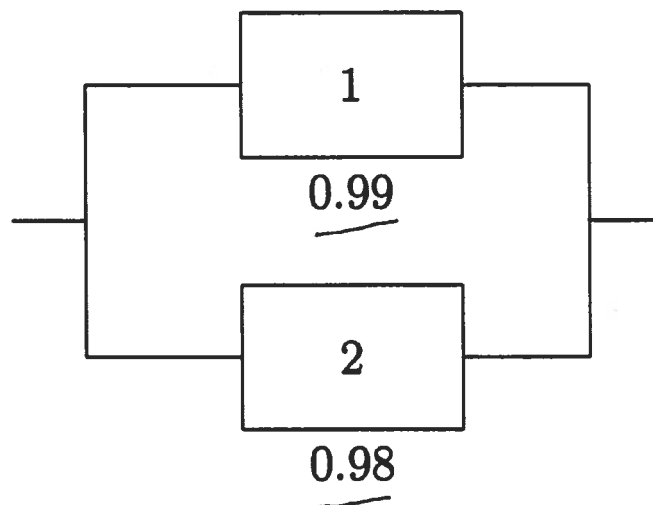
If A and B are independent, then $P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A)P(B) = \frac{P(A \cap B)}{P(A \cap B)}$
independent.

$$\boxed{P(A \cap B) = P(B)P(A)} \quad (2)$$

Note: we do not use Venn diagrams to visualize independence. If asked to show two events are independent, you may use either applicable formula above.

Example 3

Recall the simplified circuit from example 2.



$$P(A) = 0.99$$

$$P(B) = 0.98$$

$$P(A \cap B) = 0.9702$$

- (a) If component 1 is operational, what is the probability that component 2 is working?
- (b) Are the two components independent?

$$a) P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.9702}{0.99} = 0.98$$

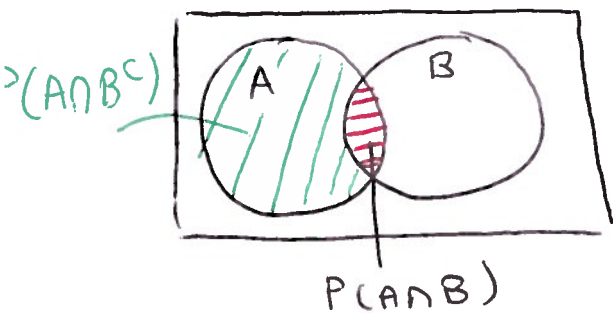
because $P(B|A) = P(B) \Rightarrow A, B$ independent.

$$P(A)P(B) = 0.98 \times 0.99 = 0.9702 = P(A \cap B)$$

If events A and B are independent then

- (a) A^c and B are also independent
- (b) A and B^c are also independent \odot^*
- (c) A^c and B^c are also independent

Proof for (b): want to show $\boxed{P(A \cap B^c) = P(A)P(B^c)}$



$$P(A) = \underline{P(A \cap B^c)} + P(A \cap B)$$

$$P(A \cap B^c) = P(A) - P(A \cap B)$$
$$= P(A) - P(A)P(B) \quad A, B \text{ independent}$$

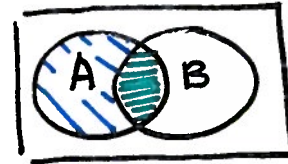
$$= P(A) [1 - P(B)]$$

$$= P(A) P(B^c) \quad \text{complement rule}$$

therefore A, B^c independent.

Example 4

Suppose A and B are independent events, where $P(A) = 0.6$ and $P(B) = 0.3$. Find $P(A \cap B^c)$.



$$P(A) - P(A \cap B) = P(A \cap B^c)$$
$$P(A) - P(A)P(B)$$

Solution:

Since A and B are independent, then A and B^c are also independent

$$P(A \cap B^c) = P(A) \times P(B^c) = 0.6 \times 0.7 = 0.42$$

Alternate method:

$$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) && \text{[Since } A \text{ and } B \text{ are independent]} \\ &= 0.6 - (0.6 \times 0.3) \\ &= 0.42 \end{aligned}$$

Last Class:

Sept. 14/16

Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

↑
given

$$P(B) > 0$$

Independence:

If A, B independent

$$\underline{P(A)} = \underline{P(A|B)}$$

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A)P(B)$$

Example 5

A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{17}{30}$. Find $P(A \cap B)$.

~~$P(A \cap B) = P(A)P(B)$~~

$P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - \underline{P(A \cap B)}$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$
 $= \frac{1}{3} + \frac{2}{5} - \frac{17}{30} = \frac{1}{6}$

we don't know
if A, B indep.

Example 6

An electronic system has 2 components operating independently of one another (i.e. one component's failure or non-failure does not affect the second component's chance of failure/non-failure). Each component has a 0.75 probability of operating properly. If the system is instead connected in series, what is the probability that the entire system operates?



$A_i =$ event component i works $i = \{1, 2\}$

$$P(\text{system works}) = P(A_1 \cap A_2) = P(A_1)P(A_2) \quad A_1, A_2 \text{ independent}$$
$$= 0.75^2 = 0.5625$$

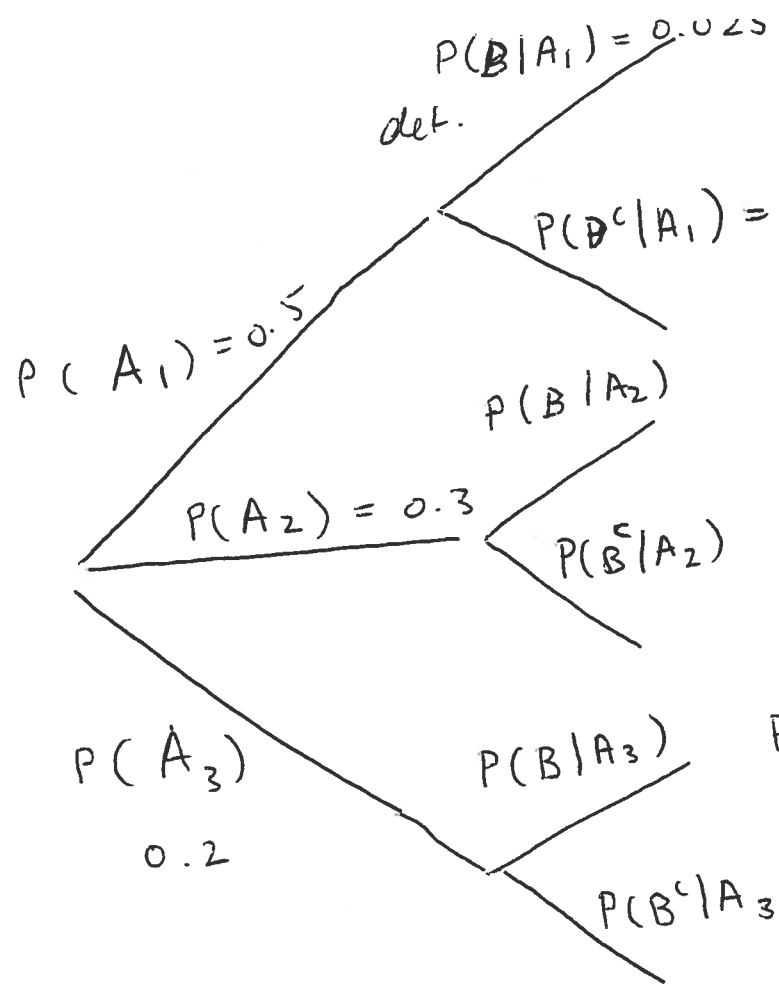
Bayes Theorem

The conditional probability of an event is a probability obtained with the additional information that some other event has already occurred. Bayes' theorem is used for calculating conditional probabilities and is best understood through an example.

Example 7

Suppose an assembly plant receives voltage regulators from three different suppliers. Of the regulators, 50% are from supplier 1, 30% from supplier 2 and 20% from supplier 3. It is known that 2.5% of supplier 1's regulators are defective, whereas the corresponding percentages for suppliers' 2 and 3 are 2% and 1% respectively.

- (a) What is the probability that a randomly selected regulator comes from supplier 1 and is defective?
- (b) What is the probability that a randomly selected regulator is defective?
- (c) If we select a regulator and find that it is defective, what is the probability it came from supplier 1? Supplier 2? Supplier 3?



$$P(A_1 \cap B) = P(B|A_1)P(A_1)$$

$$= 0.025 \times 0.5 = \underline{0.0125}$$

$$P(B \cap A_2) = P(B|A_2)P(A_2) = 0.006$$

$$P(B \cap A_3) = P(B|A_3)P(A_3) = 0.002$$

Let $A_i = \{ \text{regulator comes from supplier } i \}$ $i = \{1, 2, 3\}$

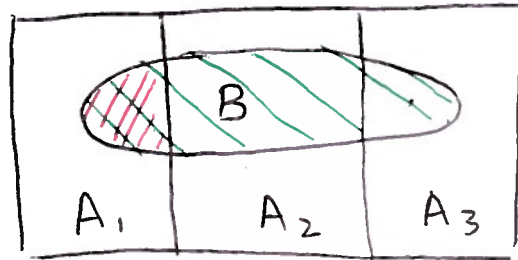
$$P(A_1) = 0.5 \quad P(A_2) = \cancel{0.3} 0.3 \quad P(A_3) = 0.2$$

$B = \text{defective}$

$$P(B | A_1) = 0.025$$

$$P(B | A_2) = 0.02$$

$$P(B | A_3) = 0.01$$



$$a) P(A_1 \cap B) = \underline{\underline{0.0125}}$$

$$b) P(B) = 0.0125 + 0.006 + 0.002 = \underline{\underline{0.0205}}$$

$$c) P(A_1 | B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.0125}{0.0205} = 0.61$$

$$P(A_2 | B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{0.006}{0.0205} = 0.29$$

$$P(A_3 | B) = 1 - 0.61 - 0.29 = 0.1$$

$$P(A_i \cap B) = P(B | A_i) P(A_i)$$

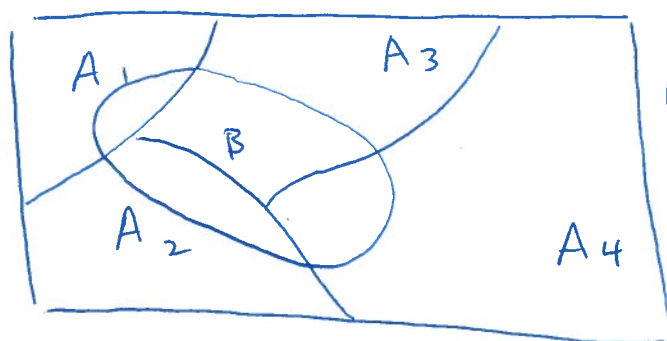
Bayes Theorem

Let A_1, A_2, \dots, A_n be disjoint events that together form the sample space S . Let B be any event from the same sample space, such that $P(B) > 0$. Then,

$$P(A_i|B) = \frac{P(A_i \cap B)}{\sum_{i=1}^n P(B|A_i)P(A_i)}$$

posterior probability

prior probability



$$P(B) = P(B|A_1)P(A_1) + \dots + P(B|A_k)P(A_k)$$

Law of Total Probability.