Physics 1003/BIT 1203
Fall 2016

Lecture 15

Rotational Kinetic Energy
Rotational Inertia
Parallel Axis Theorem

Definition of Torque

- The torque is the cross product of the distance from the centre of rotation and the force applied at that point.

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

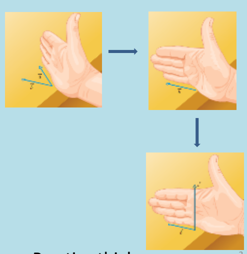
Displacement must come first, as the cross product is not commutative

https://en.wikipedia.org/wiki/Torque

The Right Hand Rule

- Point fingers on right hand in direction of vector **a**
- Curl fingers in direction of vector **b**
- Thumb indicates direction of the cross product – vector **c**

$$\vec{c} = \vec{a} \times \vec{b}$$



Practice this!

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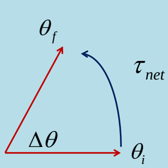
$$|\tau_{net}| = rF \sin \phi$$

- The SI unit of Torque is the Newton.metre, N.m
- Note that it is ALWAYS written as Newton.metre or N.m to identify it as a rotational unit.
- Work may be expressed in either Joules or N.m.
  - (I suggest always using Joules to avoid confusion)
  - Torque and work are not equivalent quantities, even though they have the same units
  - This is because torque is a vector quantity and work/energy is a scalar

Work Done By Constant Torque

- The work done by a constant torque can be defined in terms of the angular displacement
- If the torque produces an angular displacement Δθ, then the work done is



$$W = \tau \Delta \theta$$

Equation 10.54

Magnitude of torque

Example: Work Done by Constant Torque

- Suppose a constant torque of 250 N.m rotates an object through 4 revolutions. Calculate the work done

First convert the angular displacement to radians

$$1\ rev = 2\pi\ radians$$
$$\Delta\theta = 4 \times 2\pi\ radians$$

$$W = \tau\Delta\theta$$
$$W = (250\ N.m)(8\pi\ radians) = 6.3 \times 10^3 Joules$$

Work Done by a Variable Torque

- Integrate the torque function with respect to the angle

$$W = \int_{\theta_1}^{\theta_1} \tau(\theta)\, d\theta$$

Similar to the work done by a varialbe force $W = \int_{x_1}^{x_1} F(x)\, dx$

- Suppose the torque required to tighten a bolt varies with the angle

$$\tau = 85\ N.m + (4.3\ N.m.rad^{-1})\ \theta$$

The torque has to overcome a constant frictional torque and gets stronger as the angular displacement increases

- How much work does it take to turn it from θ = 0 to θ = 4π (2 complete turns)?

- Integrate the torque with respect to the angle

$$W = \int_{0}^{4\pi} 85\ N.m + (4.3\ N.m.rad^{-1})\ \theta\ d\theta$$

$$W = \left[85 \times \theta + (4.3\ N.m.rad^{-1})\frac{\theta^2}{2}\right]_0^{4\pi}$$

$$W = 85 \times 4\pi + (4.3\ N.m.rad^{-1})\frac{4\pi^2}{2} = 1153\ \text{J}$$

$$W = 1200\ J\ to\ 2\ s.f.$$

- If the rotation was from 0 to π/4 radians, and the torque function was

$$\tau = 2.0\theta$$

- Calculate the work done.

$$W = \int_{0}^{\pi/4} 2.0\theta\ d\theta$$

$$W = 2.0\left[\frac{\theta^2}{2}\right]_0$$

$$W = \left[(\frac{\pi}{4})^2\right] = 0.62\ \text{J}$$

Work-Kinetic Energy Theorem

- The rotational work done on a system is equal to the change in the rotational kinetic energy of the system

$$W = K_f - K_i = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2$$

Eqn. 10-52

  – This is exactly analogous to the work-energy theorem for translational motion

$$W_{tr} = K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Example: Work-Kinetic Energy Theorem

- Suppose a torque does 1200 J of work on a rotating system (I = 43 kg.m^2), and the initial angular speed was 0. Find the final angular speed.

$$W = \frac{1}{2}\text{I}\omega_f^2 - \frac{1}{2}\text{I}\omega_i^2$$

$$W = \frac{1}{2}I\omega_f^2$$

$$\frac{2W}{I} = \omega_f^2$$

- Rearranging

$$\omega_f = \sqrt{\frac{2W}{I}}$$

$$\omega_f = \sqrt{\frac{2400J}{43\ kg.m^2}} = 7.5\ rad\ /sec\ to\ 2\ s.f.$$

Rotational Inertia

- In all of the work on rotational motion, we need to calculate I, the rotational inertia (also called moment of inertia) to be able to use

$$\tau = I\alpha$$
$$W = \tau\Delta\theta$$
$$K_{rot} = \frac{1}{2}I\omega^2$$
$$L = I\omega$$

- So, how do we calculate I?

Rotational Inertia

- The rotational inertia for a rotating body is given by the expression

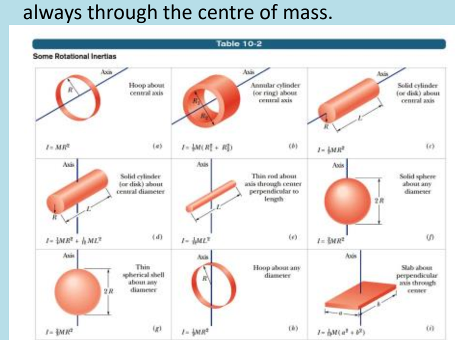
$$I = \sum m_i r_i^2 \qquad \text{Eqn. 10-34}$$

- Which for a continuous mass distribution can be written as

$$I = \int r^2 dm \qquad \text{Eqn. 10-35}$$

- It is the equivalent quantity in rotational moment to the mass for linear motion

- For well defined geometric shapes, the integrals can be evaluated to give general expressions. The axis is always through the centre of mass.



Rotational Inertia Calculation for an Object Composed of Discrete Particles

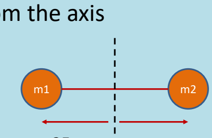
- This is where we use the summation definition

$$I = \sum m_i r_i^2$$

- The distance r is always measured relative to the centre of rotation.
- If you can set the centre of your coordinate system there, it makes things simpler

Rotational Inertia of a Two Body System

- Suppose we have two particles of mass 250g, rotating about a common axis, each 25 cm from the axis



$$I = \sum m_i r_i^2$$

$$I = m_1 r_1^2 + m_2 r_2^2$$

$$I = (0.25\ kg)(0.25\ m)^2 + (0.25\ kg)(0.25\ m)^2$$

$$I = 3.1 \times 10^{-2} kg.m^2$$

Rotational Inertia for a Continuous Mass Distribution

- For many objects, we know the shape and the density, but not the coordinates of every individual particle
  - This is where we integrate over the mass of the object

$$I = \int r^2 dm$$

- The integral takes into account the shape of the object

- As for centre of mass calculations, it is more convenient to integrate the mass distribution over a volume

$$dm = \rho dV$$

$$I = \int r^2 \rho dV$$

- This equation is valid for density variable over the volume. If the density is constant

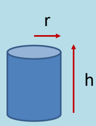
$$I = \rho \int r^2 dV$$

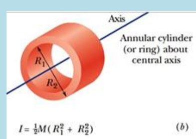
The Annular Cylinder

- We consider this cylinder to consist of many concentric shells, each with radius r and thickness dr.
- The volume of each cylinder is

$$dV = 2\pi rhdr$$

Circumference of circle



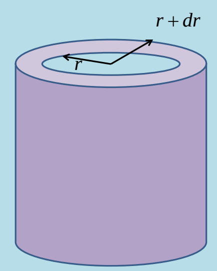


- The mass of each cylinder is

$$dm = \rho dV$$

$$dm = \rho \times 2\pi rhdr$$

Assume constant density



- The moment of inertia of each cylinder is

$$dI = r^2 dm$$

$$dI = r^2 dm$$

$$dI = r^2 \times 2\pi \rho hrdr$$

$$dI = 2\pi \rho hr^3 dr$$

We now integrate from r = R_1 to r = R_2 (the outer radius of the solid cylinder), to find the total moment of inertia

$$I = \int_{R_1}^{R_2} 2\pi \rho hr^3 dr$$

$$I = 2\pi \rho h \int_{R_1}^{R_2} r^3 dr$$

Constant, can be taken outside the integral

$$I = 2\pi \rho h \left[\frac{r^4}{4} \right]_{R_1}^{R_2}$$

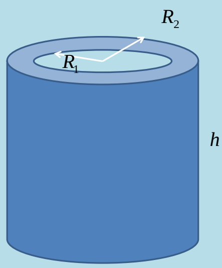
$$I = \pi \rho h \left[\frac{R_2^4}{2} - \frac{R_1^4}{2} \right]$$

$$I = \pi\rho h\left[\frac{R_2^4}{2} - \frac{R_1^4}{2}\right]$$

Want to simplify this term

$$\rho = \frac{mass}{volume} = \frac{M}{V}$$

$$V = \pi R_2^2 h - \pi R_1^2 h$$



$$I = \pi\rho h\left[\frac{R_2^4}{2} - \frac{R_1^4}{2}\right]$$

$$I = \frac{\pi M h}{V}\left[\frac{R_2^4}{2} - \frac{R_1^4}{2}\right]$$ Replace density

$$I = \frac{\pi M h}{\pi h\left[R_2^2 - R_1^2\right]}\left[\frac{R_2^4}{2} - \frac{R_1^4}{2}\right]$$ Replace volume

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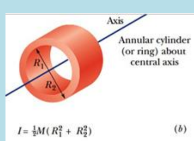
$$I = \frac{1}{2}\frac{M}{\left[R_2^2 - R_1^2\right]}\left[R_2^4 - R_1^4\right]$$

$$(a+b)(a-b) = a^2 - b^2$$

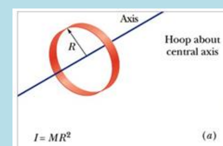
$$I = \frac{1}{2}\frac{M}{\left[R_2^2 - R_1^2\right]}\left[R_2^2 - R_1^2\right]\left[R_2^2 + R_1^2\right]$$

$$I = \frac{1}{2}M\left[R_2^2 + R_1^2\right]$$



Hoop or Thin Walled Cylinder

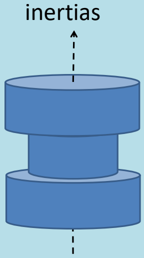
- Cylindrical shell
- Rotational axis about the central axis
- This is the same as annular cylinder, except that now $R_1 \approx R_2$



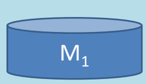
$$I = \frac{1}{2}M\left[R^2 + R^2\right]$$

$$I = MR^2$$

- If several geometries are rotating about the same axis, then the combined rotational inertia is the sum of the individual rotational inertias

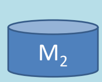


$$I_1 = \frac{1}{2}M_1R_1^2$$



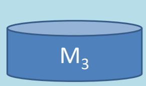
+

$$I_2 = \frac{1}{2}M_2R_2^2$$



+

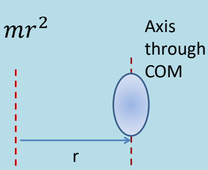
$$I_3 = \frac{1}{2}M_3R_3^2$$



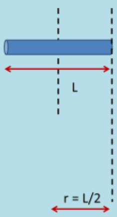
Parallel Axis Theorem

- If the object is rotating about an axis which is not through the centre of mass, but is parallel to the centre of mass, then the moment of inertia is calculated as

$$I_{total} = I_{com} + mr^2$$



- Suppose a thin rod is rotating about one end
- Look up the value of I if it rotates about the COM $I_{com} = \frac{1}{12}ML^2$



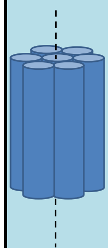
$$I_{total} = I_{com} + mr^2$$

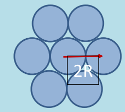
$$I_{total} = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2$$

$$I_{total} = \frac{1}{12}ML^2 + \frac{1}{4}ML^2$$

$$I_{total} = \frac{1}{3}ML^2$$

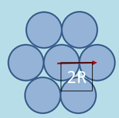
- Suppose these 7 rods, each mass M and radius R are rotating about the central axis. What is the total moment of inertia?





Central Rod $I_{com} = \frac{1}{2}MR^2$

6 outer rods, using parallel axis theorem



$$I_{total} = I_{com} + Mr^2$$

$$I_{total} = \frac{1}{2}MR^2 + M(2R)^2$$

$$I_{total} = \frac{1}{2}MR^2 + 4MR^2$$

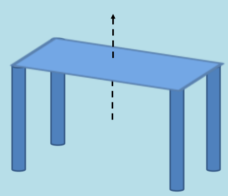
$$I_{outer} = 6 \times \left(\frac{1}{2}MR^2 + 4MR^2\right)$$

$$I_{outer} = 27MR^2$$

$$I = I_{inner} + I_{outer} = \frac{1}{2}MR^2 + 27MR^2 = 27.5MR^2$$

- Since moments of inertia are integrals, we can also break down complex shapes into easier combinations of simple geometries, where the moments of inertia are known

Table could be a rectangular slab and 4 cylinders



Rotation of a rectangular slab through the centre of mass

Rotation of 4 cylinders calculated using the parallel axis theorem

Moment of Inertia and the Rotation Axis

- The value of the moment of inertia depends on the distance of the mass away from the axis of rotation, and is different for different rotation axes (unlike mass which is always constant)