University of Ottawa
CSI 2101 – Midterm Test Solution
Instructor: Lucia Moura

February 18, 2011
4:00 pm
Duration: 1:20 hs

Closed book, no calculators

Last name: ______________________

First name: ______________________

Student number: ______________________

There are 4 questions and 100 marks total.

This exam paper should have 11 pages,
including this cover page.

1 – Predicate logic	/ 30
2 – Inference rules	/ 20
3 – Proof Methods	/ 20
4 – Number Theory	/ 30
Total	/ 100

1 Predicate logic — 30 points

Part A — 12 points

Circle true or false

$\forall n \exists m (n - m = 0)$, where the domain is the set of natural numbers.	**[true]**	[false]
$\exists n \forall m (n - m = 0)$, where the domain is the set of natural numbers.	[true]	**[false]**
The following are logically equivalent: $\neg(p \wedge \neg q)$ and $(p \rightarrow q)$	**[true]**	[false]
The following are logically equivalent: $\exists x \neg Q(x)$ and $\neg \forall x \neg Q(x)$	[true]	**[false]**
The following are logically equivalent: $\neg \forall x \exists y P(x, y)$ and $\exists x \forall y \neg P(x, y)$	**[true]**	[false]
Consider the universe of discourse to be the set $\{1, 2, 3\}$, and $Q(x, y) =$ "$y \geq x$". Then $\exists y \forall x Q(x, y)$ is true.	**[true]**	[false]

Part B — 18 points

Assume the universe of discourse to be all people, and the following statements:

$P(x)$: "x is a professor."
$I(x)$: "x is ignorant."
$V(x)$: "x is vain."

Translate each of the following phrases using quantifiers, logical connectives and $P(x)$, $I(x)$ and $V(x)$.

	phrase in English	logical statement
1.	No professors are ignorant.	$\neg \exists x (P(x) \wedge I(x)) \equiv \forall x (P(x) \rightarrow \neg I(x))$
2.	All ignorant people are vain.	$\forall x (I(x) \rightarrow V(x))$
3.	Some professors are vain.	$\exists x (P(x) \wedge V(x))$

For each of the English phrases above, write its negation in English.
Then, translate each of the negated English phrases into predicate logic; make sure the logical statement you write has the $\neg$ connectives applied to individual propositional functions only ($P(x)$, $I(x)$ or $V(x)$), that is, no $\neg$ connective is outside quantifiers or outside expressions involving other connectives.

	negated English prases	logical statement
1.	Some professors are ignorant.	$\exists x (P(x) \wedge I(x))$
2.	Some ignorant people are not vain.	$\exists x (I(x) \wedge \neg V(x))$
3.	No professors are vain. (or) All professors are not vain.	$\neg \exists (P(x) \wedge V(x))$ $\equiv \forall x (P(x) \rightarrow \neg V(x))$

2 Inference rules — 20 points

Part A — 10 points Use a formal proof and rules of inference to show that if the premises $\forall x(P(x) \rightarrow Q(x))$, $\forall x(Q(x) \rightarrow R(x))$ and $\neg R(a)$, where a is in the domain, are true, then the conclusion $\neg P(a)$ is true.

Formal Proof:

Step	Reason
1. $\forall x(P(x) \rightarrow Q(x))$	Hypothesis
2. $P(a) \rightarrow Q(a)$	Universal instantiation of (1)
3. $\forall x(Q(x) \rightarrow R(x))$	Hypothesis
4. $Q(a) \rightarrow R(a)$	Universal instantiation of (3)
5. $P(a) \rightarrow R(a)$	Hypothetical syllogism of (2) + (4)
6. $\neg R(a)$	Hypothesis
7. $\neg P(a)$	Modus tollens of (5) + (6)
8.	
9.	

Part B — 10 points Determine if the argument is correct or not using the steps below.

The premises:
"Every person attends an inexpensive school or lives in a basement."
"Every person is smart or attends an expensive school."

yield the conclusion that

"Each person is smart or lives in a basement."

- (4 points) Define the required predicates needed in the next part to express the premises and conclusions in predicate logic.

  Note: the domain of discourse considered is all people who attend some school.

  $E(x)$: x attends an expensive school
  $B(x)$: x lives in a basement
  $S(x)$: x is smart

- (4 points) Write the rule that expresses that the premises lead to the conclusion. That is, express the premises and conclusions in predicate logic in the format of a rule of inference.

$$\begin{array}{rl} & \forall x(\neg E(x) \vee B(x)) \\ & \forall x(S(x) \vee E(x)) \\ \hline \therefore & \forall x(S(x) \vee B(x)) \end{array}$$

- (2 points) Is the argument above correct?

  Yes. We can prove this by using universal instantiation, followed by resolution, and finally universal generalization, which gives the result.

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3 Proof Methods — 20 points

For this question you will need the definitions of rational and irrational numbers, seen in class.

DEFINITION: A real number r is **rational** if there exists integers a and b with $b \neq 0$ such that $r = a/b$. A real number that is not rational is called **irrational**.

Part A — 10 points Use a proof by contraposition to prove that if x is irrational, then $1/x$ is irrational.

Assume $1/x$ is irrational. Then $1/x = a/b$ for some integers a, b with $b \neq 0$.

We have that $a \neq 0$: otherwise, $1/x = 0/b = 0$, which is impossible.

Thus, $x = b/a$ for some integers a, b with $a \neq 0$. By definition, this means that x is rational.

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Part B — 10 points Prove the following:

For any integer number n, if $3n+2$ is even then n is even.

using

B1 (5 points) a proof by contraposition.

Let n be an integer, and assume n is odd. Then there exists an integer k such that:

$$n = 2k + 1.$$

This gives that:

$$\begin{aligned} 3n + 2 &= 3(2k+1) + 2 \\ &= 6k + 5 \\ &= 2(3k+2) + 1 \end{aligned}$$

Thus, $3n+2$ is odd.

B2 (5 points) a proof by contradiction.

Assume that for some integer n, $3n+2$ is even but n is odd. Thus, there exists an integer k such that:

$$n = 2k + 1.$$

This gives that:

$$\begin{aligned} 3n + 2 &= 3(2k+1) \\ &= 6k + 5 \\ &= 2(3k+2) + 1 \end{aligned}$$

This implies that $3n+2$ is odd, which contradicts the hypothesis.

4 Number Theory — 30 points

Part A — 10 points

- Give the prime factorisation of 10! (10 factorial)

$$\begin{aligned} 10! &= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= (2^1 \times 5^1) \times 3^2 \times 2^3 \times 7^1 \times (2^1 \times 3^1) \times 5^1 \times 2^2 \times 3^1 \times 2^1 \\ &= 2^8 \times 3^4 \times 5^2 \times 7 \end{aligned}$$

- Determine whether each pair of integers is relatively prime:

21, 34 (X) yes ( ) no

17, 34 ( ) yes (X) no

7, 28 ( ) yes (X) no

25,49 (X) yes ( ) no

Part B — 5 points

If the **product** of two numbers is $2^7 3^8 5^2 7^{11}$,
and their **greatest common divisor** is $2^3 3^4 5$,
what is their **least common multiple** ?

Let a and b be the two numbers. Then:

$$\text{lcm}(a,b) = \frac{ab}{\gcd(a,b)} = \frac{2^7 3^8 5^2 7^{11}}{2^3 3^4 5} = 2^4 3^4 5^1 7^{11}.$$

Part C — 5 points

Use the Euclidean algorithm to calculate $gcd(201, 111)$. Show each step.

$$\begin{aligned}
201 &= 1 \cdot 111 + 90 \\
111 &= 1 \cdot 90 + 21 \\
90 &= 4 \cdot 21 + 6 \\
11 &= 3 \cdot 6 + \mathbf{3} \\
5 &= 2 \cdot 3 + 0
\end{aligned}$$

Thus, $\gcd(201, 111) = 3$.

Part D — 10 points Prove the following result.

> If a, b, c, d, and m are integers with $m \geq 2$, with $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then
> $$a - c \equiv b - d \pmod{m}.$$

Note: You cannot use results that are very similar to this one; for example, you cannot use the fact that the given hypotheses imply $a + c \equiv b + d \pmod{m}$.
Instead, make sure you stick to using the definitions of congruences modulo m and basic divisibility properties.

As $a \equiv b \pmod{m}$, we have that $m|(a-b)$, i.e. $a - b = km$ for some integer k.

Similarly, since $c \equiv d \pmod{m}$, then $m|(c-d)$, and thus $c - d = k'm$ for some integer k'.

This gives that $(a-b) - (c-d) = (k-k')m$. By rearranging terms, we have that:

$$(a-b) - (c-d) = (a-c) - (b-d).$$

Thus, $m|(a-c) - (b-d)$, so:

$$(a-c) \equiv (b-d) \pmod{m}.$$

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Appendix 1: Logical Equivalences

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

TABLE 8 Logical Equivalences Involving Biconditionals.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

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Appendix 2: Rules of Inference

inference rule	tautology	name
p $p \rightarrow q$ $\therefore q$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens (mode that affirms)
$\neg q$ $p \rightarrow q$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens (mode that denies)
$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge (\neg p)) \rightarrow q$	disjunctive syllogism

p $\therefore p \vee q$	$p \rightarrow (p \vee q)$	addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \rightarrow (p \wedge q)$	conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$	resolution

Rule of Inference	Name
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existencial generalization