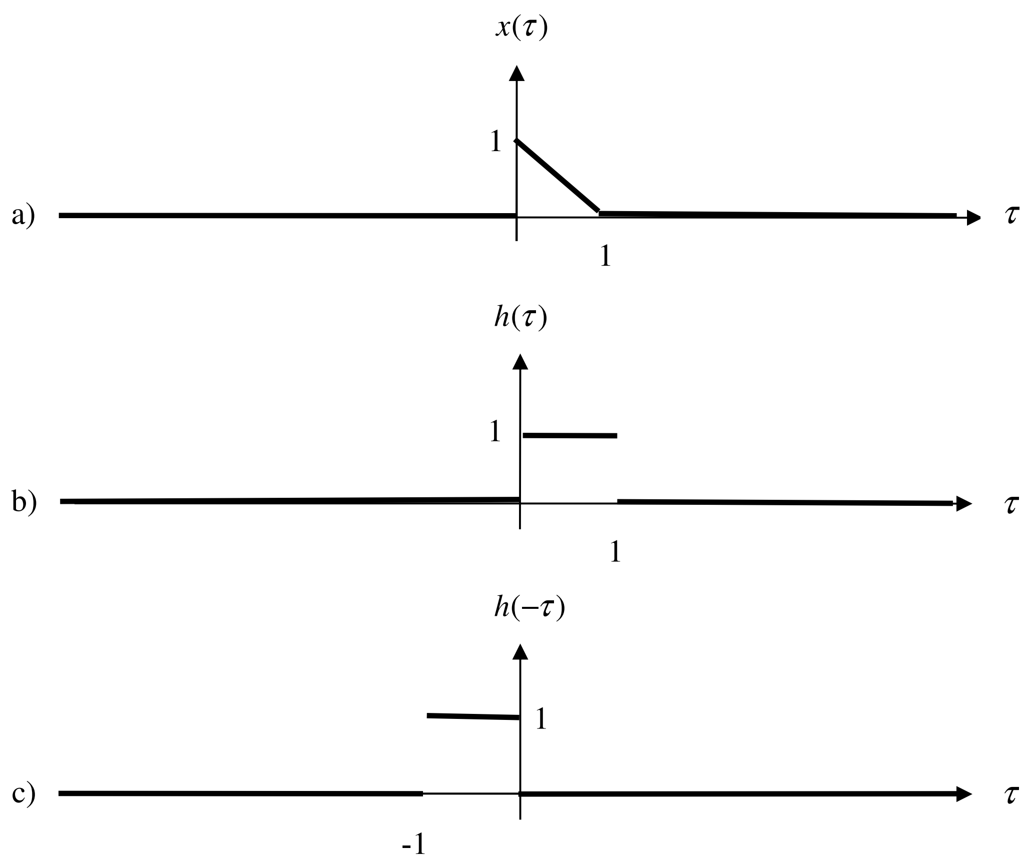
Question 1:

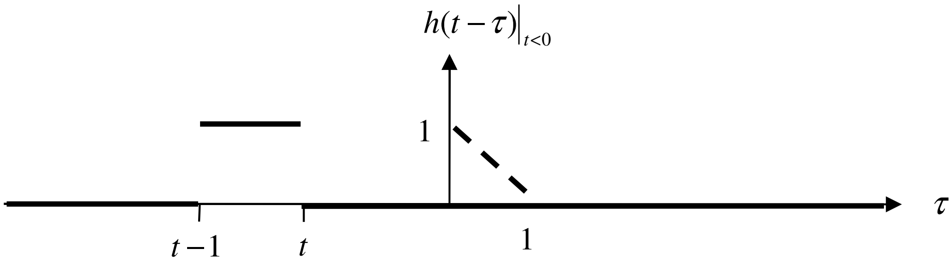
Consider a continuous-time LTI system which has impulse response of $h(t) = u(t) - u(t-1)$. If $x(t) = (1-t)\{u(t) - u(t-1)\}$ is applied at the input of the system, evaluate the output $y(t)$ of the system using convolution integral $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ as follows:

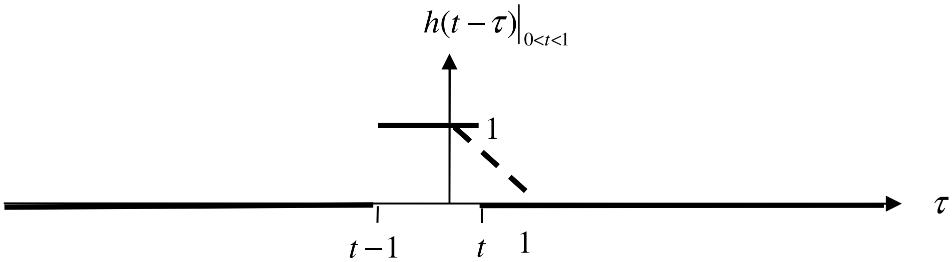
a) Draw $x(\tau)$ and $h(t-\tau)$ for different intervals of "t".
b) Write the expressions for calculating the output $y(t)$ for the intervals of "t" indicated in part (a). Do not calculate the result of integrals.

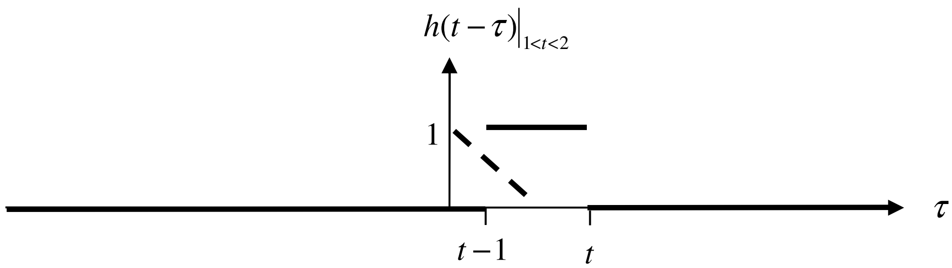
Solution 1a: $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

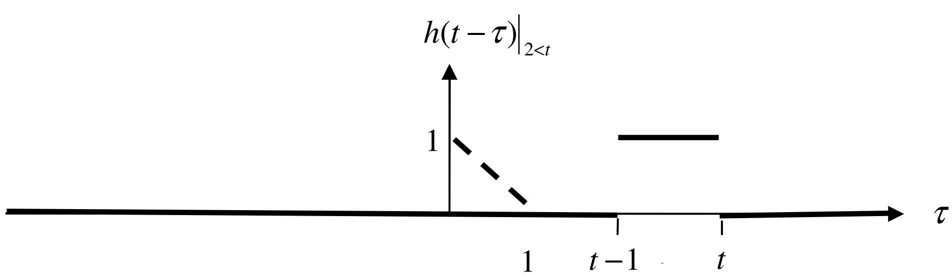
Therefore, we draw $x(\tau)$ and $h(t-\tau)$ and perform the convolution:



d)


e)


f)


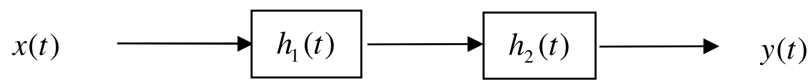
g)


Solution 1b: Based on the overlap of $x(\tau)$ in figure "a" and $h(t-\tau)$ in figures "d", "e", "f" and "g", $y(t)$ can be calculated for various regions as shown below:

$$y(t)=\begin{cases} 0 & t<0 \quad Fig.\ d) \\ \int_0^t (1-\tau)d\tau = \left[\tau-\frac{\tau^2}{2}\right]_0^t = t-\frac{t^2}{2} & 0<t<1 \quad Fig.\ e) \\ \int_{t-1}^1 (1-\tau)d\tau = \left[\tau-\frac{\tau^2}{2}\right]_{t-1}^1 = 2-2t+\frac{t^2}{2} & 1<t<2 \quad Fig.\ f) \\ 0 & 2<t \quad Fig.\ g) \end{cases}$$

Question 2:

Consider the interconnection of two continuous-time LTI systems, as depicted in the figure. The impulse responses of these systems are $h_1(t) = \delta(t-1) + \delta(t+1)$, $h_2(t) = 2^t u(t)$,



a) Find the impulse response of the overall system.
b) Determine whether or not each of the impulse responses $h_1(t)$ and $h_2(t)$ are causal, memory-less and stable. Justify your answer.

Solution 2a: $h(t) = h_1(t) * h_2(t) =$

$$h(t) = \int_{-\infty}^{\infty} [\delta(\tau-1) + \delta(\tau+1)] \times 2^{t-\tau} u(t-\tau) d\tau = \int_{-\infty}^{\infty} \delta(\tau-1) \times 2^{t-\tau} u(t-\tau) d\tau + \int_{-\infty}^{\infty} \delta(\tau+1) \times 2^{t-\tau} u(t-\tau) d\tau$$

$$h(t) = \int_{-\infty}^{\infty} \delta(\tau-1) \times 2^{t-\tau} u(t-\tau) d\tau + \int_{-\infty}^{\infty} \delta(\tau+1) \times 2^{t-\tau} u(t-\tau) d\tau =$$

$$h(t) = \int_{-\infty}^{\infty} \delta(\tau-1) \times 2^{t-1} u(t-1) d\tau + \int_{-\infty}^{\infty} \delta(\tau+1) \times 2^{t+1} u(t+1) d\tau = 2^{t-1} \int_{1}^{\infty} \delta(\tau-1) d\tau + 2^{t+1} \int_{-1}^{\infty} \delta(\tau+1) d\tau =$$

$$h(t) = 2^{t-1} u(t-1) + 2^{t+1} u(t+1)$$

Solution 2b:

If the impulse response of a system is zero for all values of $t < 0$, then the system is causal. Therefore:

Since $h_1(t) = \delta(t-1) + \delta(t+1)$, which is non-zero at $t = -1$ is non-causal.
Since $h_2(t) = 2^t u(t)$ is zero for all values of $t < 0$ (due to $u(t)$), therefore $h_2(t)$ is causal.

A system is memory-less if the impulse response is in the form of $K\delta(t)$ and obviously both $h_1(t)$ and $h_2(t)$ are not in this form and they have memory.

A system is stable if the impulse response satisfies, $\int_{-\infty}^{\infty} |h(\tau| d\tau < \infty$. Therefore:

$$\int_{-\infty}^{\infty} |h_1(\tau)| d\tau = \int_{-\infty}^{\infty} |\delta(t-1) + \delta(t+1)| d\tau = 2 < \infty$$ and $h_1(t)$ is stable.

$$\int_{-\infty}^{\infty} |h_2(\tau)| d\tau = \int_{-\infty}^{\infty} |2^{\tau} u(\tau)| d\tau = \int_{0}^{\infty} |2^{\tau}| d\tau = \infty$$ and $h_2(t)$ is unstable.

Question 3:

Consider a continuous-time LTI system which has input of $x(t)$ and output of $y(t) = 2x(t-2) + x(t-1) + x(t+1) + 2x(t+2)$.

a- Evaluate Fourier series coefficients a_k of $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-5k)$.

b- Using a_k and the properties of Fourier series, evaluate the Fourier series coefficients b_k of the output $y(t)$. Simplify b_k and show the result as weighted sum of cosine functions.

Solution 3a:

The signal $x(t)$ has fundamental period of $T_0 = 5$ and fundamental frequency of $\omega_0 = 2\pi/5$. Note that in each period there is a unit impulse function.

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{5}\int_{-2.5}^{2.5}\delta(t)e^{-jk(2\pi/5)t}dt = \frac{1}{5}\int_{-2.5}^{2.5}\delta(t)e^{-jk(2\pi/5)0}dt = \frac{1}{5}\int_{-2.5}^{2.5}\delta(t)dt = \frac{1}{5}$$

$a_k = \frac{1}{5}$ for all integer values of k.

Solution 3b:

From properties, we have $x(t) \leftrightarrow a_k$, then $x(t-t_0) \leftrightarrow a_k e^{-jk(2\pi/T)t_0}$**.** By considering the linearity property as well, we have:

$$2x(t-2) + x(t-1) + x(t+1) + 2x(t+2) \leftrightarrow 2a_k e^{-jk(2\pi/5)2} + a_k e^{-jk(2\pi/5)1} + a_k e^{jk(2\pi/5)1} + 2a_k e^{jk(2\pi/5)2}$$

$$2x(t-2) + x(t-1) + x(t+1) + 2x(t+2) \leftrightarrow 2a_k\left(e^{-jk(2\pi/5)2} + e^{jk(2\pi/5)2}\right) + a_k\left(e^{-jk(2\pi/5)} + e^{jk(2\pi/5)}\right)$$

$$2x(t-2) + x(t-1) + x(t+1) + 2x(t+2) \leftrightarrow 4a_k\cos\frac{4\pi k}{5} + 2a_k\cos\frac{2\pi k}{5}$$

$$2x(t-2) + x(t-1) + x(t+1) + 2x(t+2) \leftrightarrow \frac{4}{5}\cos\frac{4\pi k}{5} + \frac{2}{5}\cos\frac{2\pi k}{5}$$

$y(t) = 2x(t-2) + x(t-1) + x(t+1) + 2x(t+2) \leftrightarrow b_k = \frac{4}{5}\cos\frac{4\pi k}{5} + \frac{2}{5}\cos\frac{2\pi k}{5}$ for all integer values of k.

Question 4)

Consider a continuous-time LTI system which has input of $x(t)$ and output of $y(t)=\int_{-\infty}^{t} x(\tau)d\tau$. The input to the system is $x(t)=2\sin(2t)$.

a- Evaluate impulse response of the system $h(t)$ and its frequency domain equivalent $H(j\omega)$.

b- Evaluate Fourier series coefficients of the input signal $x(t)$ and output signal $y(t)$.

Solution 4a:

$$h(t)=y(t)\big|_{x(t)=\delta(t)}=\int_{-\infty}^{t}\delta(\tau)d\tau=u(t)$$

$$H(j\omega)=\int_{-\infty}^{+\infty}h(\tau)e^{-j\omega\tau}d\tau=\int_{-\infty}^{+\infty}u(\tau)e^{-j\omega\tau}d\tau=\int_{0}^{+\infty}e^{-j\omega\tau}d\tau=\left[\frac{e^{-j\omega\tau}}{-j\omega}\right]_{0}^{\infty}=\frac{1}{j\omega}$$

Solution 4b:

The fundamental frequency of $x(t)=2\sin(2t)$ is $\omega_0=2$ and fundamental period would be $T=\frac{2\pi}{\omega_0}=\frac{2\pi}{2}=\pi$.

$$x(t)=2\sin(2t)=2\left(\frac{e^{j2t}-e^{-j2t}}{2j}\right)=\frac{1}{j}\left(e^{j2t}-e^{-j2t}\right)$$

$$x(t)=\frac{1}{j}\left(e^{j2t}-e^{-j2t}\right)=je^{-j2t}-je^{j2t}$$

Comparing above equation with the Fourier series expansion of

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk(2\pi/T)t}=\sum_{k=-\infty}^{\infty}a_k e^{jk2t}$$

We have: $a_{-1}=j$, $a_1=-j$ and $a_k=0$ for $k\neq\pm1$

Fourier series coefficients of $y(t)$ are

$$b_k=a_k H(jk\omega_0)=a_k H(j2k)=a_k\frac{1}{j2k}$$

$$b_k=-\frac{ja_k}{2k}\Rightarrow b_{-1}=-\frac{1}{2},\quad b_1=-\frac{1}{2}\ \text{and}\ b_k=0\ \text{for}\ k\neq\pm1$$