STAT 3502
Solutions to ASSIGNMENT 3

1. **a.** Let $N(t)$ be the number of earthquakes in this region at or prior to time t. We are given that $\{N(t), t \geq 0\}$ is a Poisson process. If we choose one year as a time unite, then $\alpha = \mathbf{E}(N(1)) = 7$. Therefore

$$\mathbf{P}(N(t) = k) = e^{-7t}\frac{(7t)^k}{k!}, \quad k = 0, 1, 2, \ldots.$$

If we now denote by p the probability of no earthquakes in one year, then

$$p = \mathbf{P}(N(1) = 0) = e^{-7} = 0.00091.$$

b. If no earthquakes occur during a given year, then that year is called a success. Let X be the number of years of the next 8 years, in which no earthquakes will occur. Then $X \sim \text{Bin}(8, p)$, with p as in part a). Thus, the required probability is

$$\mathbf{P}(X = 3) = \binom{8}{3}(0.00091)^3(1 - 0.00091)^5 = 4.2 \times 10^{-8}.$$

2. **a.** The probability of interest is

$$\mathbf{P}(X > 20) = \int_{20}^{\infty} \frac{10}{x^2}dx = \frac{-10}{x}\Big|_{20}^{\infty} = 0 - (-1/2) = 1/2.$$

b. Using the formula $F(x) = \int_{-\infty}^{x} f(t)\,dt$, we obtain that

for $x < 10$, $\quad F(x) = 0,$ $\qquad$ and for $x \geq 10$, $\quad F(x) = \int_{10}^{x} \frac{10}{y^2}dy = 1 - \frac{10}{x}.$

c. Each of the 6 selected devices will function for at least 15 hours with probability $p = P(X \geq 15) = 1 - F(15) = \frac{10}{15} = 2/3$. The number Y of devices (among the 6 selected) that function for at least 15 hours has binomial distribution with parameters $n = 6$ and $p = 2/3$. Hence the desired probability is

$$\begin{aligned}
\mathbf{P}(Y \geq 3) &= 1 - \mathbf{P}(Y \leq 2) \\
&= 1 - [\mathbf{P}(Y = 0) + \mathbf{P}(Y = 1) + \mathbf{P}(Y = 2)] \\
&= 1 - [(1/3)^6 + 6(2/3)(1/3)^5 + 15(2/3)^2(1/3)^4] = 0.9.
\end{aligned}$$

d. The average lifetime (with $\log x$ denoting the natural logarithm of x) is

$$\mathbf{E}(X) = \int_{10}^{\infty} x\,\frac{10}{x^2}\,dx = 10\int_{10}^{\infty}\frac{dx}{x} = \log(\infty) - \log(10) = \infty - 10 = \infty.$$

3. We have

$$\mathbf{E}(3 - 4X) = 3 - 4\mathbf{E}(X) = 3 - 4\int_{-1}^{2}\frac{x^3}{3}\,dx = 3 - 4(5/4) = -2;$$

$$\mathbf{Var}(3 - 4X) = 16\mathbf{Var}(X) = 16\left(\mathbf{E}(X^2) - (\mathbf{E}(X))^2\right) = 16\left(\int_{-1}^{2}\frac{x^4}{3}\,dx - (5/4)^2\right)$$

$$= 16\left(\frac{33}{15} - \frac{25}{16}\right) = \frac{51}{5}.$$

4. Denote by X the weekly volume of sales. We want to find x such that $\mathbf{P}(X > x) = 0.1$. That is, we are looking for the 90th percentile of X. For $0 < x < 1$,

$$\mathbf{P}(X > x) = \int_x^1 5(1-y)^4 dy = (1-x)^5.$$

So x must satisfy $(1-x)^5 = 0.1$ or equivalently $(1-x) = (0.1)^{1/5} = 0.631$ and hence $x = 0.369$.

5. By assumption, the random variable X, the annual rainfall for a given year, has normal distribution with $\mu = 40$ and $\sigma = 4$. Using independence and Table A.3 from the textbook, the desired probability is

$$\left[\mathbf{P}(X \leq 50)\right]^{10} = \left[\mathbf{P}\left(\frac{X-40}{4} \leq \frac{50-40}{4}\right)\right]^{10} = \left[\mathbf{P}(Z \leq 2.5)\right]^{10} = (0.9938)^{10} = 0.94.$$

6. Let X denote the lifetime of Jones' used TV. Then X has an exponential distribution with parameter $\lambda = 1/5$ and cdf $F(x) = 1 - e^{-x/5}$ for $x \geq 0$. Hence, the probability of interest is

$$\begin{aligned}\mathbf{P}(X > 3+6|X > 3) &= \frac{\mathbf{P}[(X > 9) \cap (X > 3)]}{P(X > 3)} = \frac{\mathbf{P}(X > 9)}{\mathbf{P}(X > 3)} = \frac{e^{-9/5}}{e^{-3/5}} = e^{-6/5} \\ &= 0.301 = P(X > 6).\end{aligned}$$

Hence the probability the used TV will be working after additional 6 years (having worked up to 3 years) is the same as the probability of working after 6 years from the beginning.

7. **a.** Denote by X the fuse's lifetime. This is an exponential random variable with parameter $\lambda = 1/2$. Its cdf is given by $F(x) = 1 - e^{-0.5x}$ for $x \geq 0$. The median satisfies $F(\tilde{\mu}) = 0.5$, and hence it can be computed from the equation

$$1 - e^{-0.5\tilde{\mu}} = 0.5.$$

From this, $e^{-0.5\tilde{\mu}} = 0.5$ or $0.5\tilde{\mu} = \log 2$. Finally, we get $\tilde{\mu} = 2(\log 2) = 1.386$.
b. We want to find x such that $1 - F(x) = 0.75$. That is, $e^{-0.5x} = 3/4$ or equivalently $0.5x = \log(4/3)$. Hence $x = 2\log(4/3) = 0.575$.

8. **a.** The waiting time T between two successive accidents has exponential distribution with parameter $\lambda = 3$ (time unit = one year) and hence the average waiting time between two successive accidents is $\mathbf{E}(T) = 1/3$ year.
b. The desired probability is

$$\mathbf{P}(T \leq 6+1|T > 6) = \frac{\mathbf{P}(6 < T \leq 7)}{P(T > 6)} = \frac{1 - e^{-3(7)} - [1 - e^{-3(6)}]}{e^{-3(6)}} = 1 - e^{-3} = \mathbf{P}(T \leq 1).$$

So the probability of an accident occurring the next year is the same as the probability of an accident occurring within any given year (regardless of what happened in the past years).

9. **a.** We have

$$h(t) = \frac{1}{\mathbf{P}(T \geq t)} \lim_{\Delta \to 0} \frac{F(t+\Delta) - F(t)}{\Delta} = \frac{F'(t)}{1 - F(t)} = \frac{f(t)}{1 - F(t)}.$$

b. If T has Weibull distribution, then for $t > 0$,

$$h(t) = \frac{\alpha\beta^\alpha t^{\alpha-1} e^{-(\beta t)^\alpha}}{e^{-(\beta t)^\alpha}} = \alpha\beta^\alpha t^{\alpha-1}$$

and hence

$$h'(t) = \alpha(\alpha-1)\beta^\alpha t^{\alpha-1} \begin{cases} > 0, \text{ i.e. } h \text{ increasing}, & \text{if } \alpha > 1, \\ < 0, \text{ i.e. } h \text{ decreasing}, & \text{if } \alpha < 1, \\ = 0, \text{ i.e. } h \text{ constant}, & \text{if } \alpha = 1. \end{cases}$$

c. The choice of $\alpha = 1$ in Weibull distribution corresponds to the pdf of exponential distribution with parameter $\lambda = \beta$, in which case $h(t) = \beta = \lambda$.

10. **a.** The fill volume X of a randomly selected can is normally distributed with $\mu = 12$ and $\sigma = 0.03$. Hence $\mathbf{P}(X < 12) = \mathbf{P}(Z < (12-12)/0.03) = \mathbf{P}(Z < 0) = 0.5$, that is, 50%.
b. We want the new μ to satisfy $\mathbf{P}(X \geq 12) = 0.99$ or equivalently $\mathbf{P}(X \leq 12) = 0.01$. By the standardization procedure we get

$$0.01 = \mathbf{P}(X \leq 12) = \mathbf{P}\left(\frac{X-\mu}{\sigma} \leq \frac{12-\mu}{\sigma}\right) = \mathbf{P}(Z \leq z), \quad z = \frac{12-\mu}{\sigma},$$

where $Z \sim N(0,1)$. From the normal table (Table A.3 in the textbook) we can read that approximately $z = -2.33$ and hence we get $(12-\mu)/\sigma = -2.33$ and therefore $\mu = 12 + \sigma(2.33) = 12 + 0.03(2.33) = 12.07$. This is the value the mean should be set to in order to have 99% of cans containing 12 oz or more.

11. The number X (among the 215 with reservations) who will show up has binomial distribution with parameters $n = 215$ and $p = 0.9$. We have $np = 193.5$ and $nq = 21.5$. Therefore the rule of thumb is satisfied, and we can use normal approximation to binomial distribution. That is, X can be considered as (nearly) normally distributed with $\mu = np = 193.5$ and $\sigma = \sqrt{npq} = \sqrt{215(0.1)(0.9)} = 4.4$. The desired probability is then (without the use of 0.5 continuity correction)

$$\mathbf{P}(X \leq 200) = \mathbf{P}\left(\frac{X-193.5}{4.4} \leq \frac{200-193.5}{4.4}\right) = \mathbf{P}(Z \leq 1.48) = 0.9306,$$

or (with the use of 0.5 continuity correction)

$$\mathbf{P}(X \leq 200) = P(Z \leq 1.59) = 0.9441.$$