MAT 2371
Fall 2016
Assignment 5
Due on Tuesday, December 6, 2016 in class.

[10]1. 10 independent observations are chosen at random from an exponential distribution with mean 1.

Calculate the probability that at least 5 of them are in the interval $(1, 3)$.

Solution. This random experiment is in fact the repeat of Bernoulli trials for $n = 10$ times with probability success of

$$p = P(1 < X < 3) = \int_1^3 e^{-x}dx = \exp(-1) - \exp(-3) = 0.3180924.$$

Therefore the probability will be

$$\sum_{k=5}^{10} \binom{10}{k} (0.3180924)^k (1 - 0.3180924)^{10-k} = 0.1830091.$$

[24]2. Let (X, Y) have the joint p.d.f.

$$f(x, y) = \begin{cases} k & \text{if } 0 < x, \ 0 < y, x + y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

where $k > 0$ is a constant.

(i) Find k.

Solution.

$$1 = k \int_0^1 \int_0^{1-x} dydx = k \int_0^1 (1 - x)dx = k/2.$$

Therefore $k = 2$.

(ii) Find $f_1(x)$ and $f_2(y)$, the marginal distributions for X and Y. Are X and Y independent?

Solution. For $x \in (0, 1)$,

$$f_1(x) = \int_0^{1-x} 2dy = 2(1 - x).$$

Therefore

$$f_1(x) = \begin{cases} 2(1-x) & \text{if } \ 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Similarly

$$f_2(y) = \begin{cases} 2(1-y) & \text{if } \ 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Since $f(x,y) \neq f_1(x)f_2(y)$ we conclude that X and Y are not independent.

(iii) Find the covariance between X and Y.

Solution. We have

$$E(X) = E(Y) = \int_0^1 2t(1-t)dt = 1/3$$

and

$$E(XY) = 2\int_0^1 \int_0^{1-x} xydydx = \int_0^1 x(1-x)^2 dx = \frac{1}{12}.$$

This gives

$$Cov(X,Y) = 1/12 - (1/3)(1/3) = \frac{-1}{36}.$$

(iv) Find $E(X|Y=y)$ and $Var(Y|X=x)$.

Solution. We have

$$f(x|y) = \begin{cases} \frac{1}{1-y} & \text{if } \ 0 < y < 1-x \\ 0 & \text{elsewhere.} \end{cases}$$

This shows

$$(X|Y=y) \sim Unif(0, 1-y).$$

Therefore we use the formulas we have for uniform distribution.

$$E(X|Y=y) = \frac{1-y}{2}.$$

Similarly

$$f(y|X=x) = \begin{cases} \frac{1}{1-x} & \text{if } \ 0 < x < 1-y \\ 0 & \text{elsewhere.} \end{cases}$$

This means

$$(Y|X=x) \sim Unif(0, 1-x).$$

Therefore

$$Var(Y|X=x) = \frac{(1-x)^2}{12}.$$

[21]3. Let (X, Y) be a random vector such that

$$f(y|x) = \begin{cases} 1/x & \text{if } \; y = 1, 2, \dots, x \\ 0 & \text{elsewhere.} \end{cases}$$

(For any positive integer x). Also let X has the following p.m.f.

$$f_1(x) = \begin{cases} 1/3 & \text{if } \; x = 1, 2, 3 \\ 0 & \text{elsewhere.} \end{cases}$$

(i) Calculate $E(Y)$.

Solution. We have

$$f(x, y) = \begin{cases} 1/(3x) & \text{if } \; y = 1, 2, \dots, x, x = 1, 2, 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Therefore for $y = 1, 2, 3$

$$f_2(y) = \sum_{x=y}^{3} \frac{1}{3x}$$

and zero otherwise. Therefore

$$f_2(1) = 11/18, f_2(2) = 5/18, f_2(3) = 2/18.$$

Therefore

$$E(Y) = 11/18 + 10/18 + 6/18 = 27/18 = 3/2.$$

(ii) Calculate $P(X = 1|Y = 2)$.

Notice that the p.m.f. maps zero to any pair (x, y) when $y > x$.

$$P(X = 1|Y = 2) = \frac{f(1, 2)}{f_2(2)} = 0.$$

(iii) Calculate $E(X|Y = 2)$.

$$E(X|Y = 2) = \sum_{x=2}^{3} xP(X = x|Y = 2) = 2P(X = 2|Y = 2) + 3P(X = 3|Y = 2).$$

We have

$$P(X = 2|Y = 2) = \frac{f(2, 2)}{f_2(2)} = \frac{\frac{1}{6}}{\frac{5}{18}} = \frac{18}{30} = \frac{3}{5}$$

and

$$P(X = 3|Y = 2) = \frac{f(3, 2)}{f_2(3)} = \frac{\frac{1}{9}}{\frac{5}{18}} = \frac{2}{5}$$

Therefore

$$E(X|Y=2) = 2(3/5) + 3(2/5) = 12/5.$$

[15]4. Three chips are selected at random without replacement from a box containing three red chips, two white chips and four black chips. Let X and Y be the number of red chips and the number of white chips respectively.

(i) Write the p.m.f. for (X, Y) and the marginal distribution for X and Y.

Solution. We have for nonnegative integers x and y,

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{4}{3-x-y}}{\binom{9}{3}}, \quad 0 \le x \le 3, \quad 0 \le y \le 2, \quad x+y \le 3$$

and zero, otherwise. It is easy to see that for $x = 0, 1, 2, 3$

$$f_1(x) = P(X=x) = \frac{\binom{3}{x}\binom{6}{3-x}}{\binom{9}{3}},$$

and zero otherwise. Similarly for $y = 0, 1, 2$,

$$f_2(y) = P(Y=y) = \frac{\binom{2}{y}\binom{7}{3-y}}{\binom{9}{3}},$$

and zero otherwise.

(ii) Find $E(Y|X=1)$.

Solution. We have

$$E(Y|X=1) = 0P(Y=0|X=1) + 1P(Y=1|X=1) + 2P(Y=2|X=1)$$
$$= P(Y=1|X=1) + 2P(Y=2|X=1).$$

On the other hand

$$P(Y=1|X=1) = \frac{f(1,1)}{f_1(1)} = \frac{\frac{2}{7}}{\frac{15}{28}} = \frac{8}{15}$$

and

$$P(Y=2|X=1) = \frac{f(1,2)}{f_1(1)} = \frac{\frac{1}{28}}{\frac{15}{28}} = \frac{1}{15}.$$

Therefore

$$E(Y|X=1) = P(Y=1|X=1) + 2P(Y=2|X=1) = \frac{8}{15} + \frac{2}{15} = \frac{10}{15} = \frac{2}{3}.$$

[15]5. Let X and Y be be two independent random variables such that $E(X) = E(Y) = 4$ and $Var(X) = Var(Y) = 2$. Define $U = 3X - 2Y + 1$ and $V = X(2Y + X)$. Calculate $E(U)$, $E(V)$ and $Var(U)$.

Solution.

$$E(U) = 3E(X) - 2E(Y) + 1 = 5$$

and

$$E(V) = 2E(XY) + E(X^2) = 32 + E(X^2).$$

Since

$$E(X^2) = Var(X) + (E(X))^2 = 2 + 16 = 18,$$

we get

$$E(V) = 32 + 18 = 50.$$

Since $Var(U) = Var(3X - 2Y)$ and

$$E(U^2) = E((3X-2Y)^2) = 9E(X^2)+4E(Y^2)-12E(XY) = 9(18)+4(18)-12(16) = 42.$$

Therefore

$$Var(U) = 42 - 25 = 17.$$

Solution. [15] 6. Let $X_1, \ldots, X_{20}$ be a sequence of i.i.d. observation from $N(5, 16)$. Calculate

(i) $P(X_1 + \ldots + X_{10} \geq X_{11} + \ldots + X_{20})$.

Solution.

Since

$$X_1 + \cdots + X_{10} - X_{11} - \cdots - X_{20} \sim N(0, 320)$$

$$P\left(X_1 + \cdots + X_{10} - X_{11} - \cdots - X_{20} > 0\right) = 0.5.$$

Notice that the result corrects for any distribution due to the symmetry property on both side of the inequality.

(ii) $P(\sum_{i=1}^{20}(-1)^{i-1}X_i > 1)$.

Solution. We have

$$\sum_{i=1}^{20}(-1)^{i-1}X_i \sim N(0, 320).$$

Therefore

$$P\left(\sum_{i=1}^{20}(-1)^{i-1}X_i > 1\right) = P\left(Z > \frac{1}{\sqrt{320}}\right) = 0.4777.$$