

Unit 4

Lesson 16

$$\begin{aligned} 1. \quad \text{a) } \vec{u} \cdot \vec{v} &= \|\vec{u}\| \|\vec{v}\| \cos \theta \\ &= (2)(3) \cos(30^\circ) \\ &\approx 5.19 \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{u} \cdot \vec{v} &= \|\vec{u}\| \|\vec{v}\| \cos \theta \\ &= (5)(3) \cos(135^\circ) \\ &\approx -10.6 \end{aligned}$$

$$\begin{aligned} 2. \quad \text{a) } \vec{u} \cdot \vec{v} &= (-1, 2) \cdot (4, 3) \\ &= (-1)(4) + (2)(3) \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{b) } \vec{u} \cdot \vec{v} &= (4, -2, 3) \cdot (0, -1, 5) \\ &= (4)(0) + (-2)(-1) + (3)(5) \\ &= 17 \end{aligned}$$

$$\begin{aligned} 3. \quad \text{a) } \vec{u} \cdot \vec{v} &= (-1, 2) \cdot (-3, -1) \\ &= (-1)(-3) + (2)(-1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} |\vec{u}| &= \sqrt{(-1)^2 + (2)^2} \\ &= \sqrt{5} \end{aligned}$$

$$\begin{aligned} |\vec{v}| &= \sqrt{(-3)^2 + (-1)^2} \\ &= \sqrt{10} \end{aligned}$$

$$\begin{aligned}\cos(\theta) &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \\ &= \frac{1}{(\sqrt{5})(\sqrt{10})} \\ &= \frac{1}{5\sqrt{2}} \\ &\approx 0.14\end{aligned}$$

$$\theta \approx \cos^{-1}(0.14)$$

$$\theta \approx 81.95$$

The angle between the two vectors is approximately 81.95° .

$$\begin{aligned}\text{b) } \vec{u} \cdot \vec{v} &= (-1, 2, 1) \cdot (4, 0, 3) \\ &= (-1)(4) + (2)(0) + (1)(3) \\ &= -1\end{aligned}$$

$$\begin{aligned}|\vec{u}| &= \sqrt{(-1)^2 + (2)^2 + (1)^2} \\ &= \sqrt{6}\end{aligned}$$

$$\begin{aligned}|\vec{v}| &= \sqrt{(4)^2 + (0)^2 + (3)^2} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

$$\begin{aligned}\cos(\theta) &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \\ &= \frac{-1}{(\sqrt{6})(5)} \\ &\approx -0.082\end{aligned}$$

$$\theta \approx \cos^{-1}(-0.082)$$

$$\theta \approx 94.7^\circ$$

The angle between the two vectors is approximately 94.7° .

4. $\vec{u} = (4, 6)$ and $\vec{v} = (1, 4)$

$$\begin{aligned}\vec{u} \cdot \vec{v} &= (4, 6) \cdot (1, 4) \\ &= (4)(1) + (6)(4) \\ &= 28\end{aligned}$$

$$\begin{aligned}|\vec{u}| &= \sqrt{(4)^2 + (6)^2} \\ &= \sqrt{52}\end{aligned}$$

$$\begin{aligned}|\vec{v}| &= \sqrt{(1)^2 + (4)^2} \\ &= \sqrt{17}\end{aligned}$$

$$\begin{aligned}\cos(\theta) &= \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \\ &= \frac{28}{(\sqrt{52})(\sqrt{17})} \\ &\approx 0.94\end{aligned}$$

$$\theta \approx \cos^{-1}(0.94)$$

$$\theta \approx 19.95^\circ$$

The angle between the two vectors is approximately 19.95° .

5. a)
$$\begin{aligned}\vec{u} \cdot (\vec{v} + \vec{w}) &= (1, -1, 2) \cdot [(1, -1, 3) + (-1, 0, 2)] \\ &= (1, -1, 2) \cdot (0, -1, 5) \\ &= (1)(0) + (-1)(-1) + (2)(5) \\ &= 11\end{aligned}$$

$$\begin{aligned}\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} &= (1, -1, 2) \cdot (1, -1, 3) + (1, -1, 2) \cdot (-1, 0, 2) \\ &= (1)(1) + (-1)(-1) + (2)(3) + (1)(-1) + (-1)(0) + (2)(2) \\ &= 11\end{aligned}$$

Observe that $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

$$\begin{aligned} \text{b) } \vec{w} \cdot \vec{w} &= (-1, 0, 2) \cdot (-1, 0, 2) \\ &= 5 \end{aligned}$$

$$\begin{aligned} |\vec{w}|^2 &= (-1)^2 + (0)^2 + (2)^2 \\ &= 5 \end{aligned}$$

Observe that $\vec{w} \cdot \vec{w} = |\vec{w}|^2$.

$$\begin{aligned} \text{c) } (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) &= [(1, -1, 2) + (1, -1, 3)] \cdot [(1, -1, 2) + (1, -1, 3)] \\ &= (2, -2, 5) \cdot (2, -2, 5) \\ &= (2)(2) + (-2)(-2) + (5)(5) \\ &= 33 \end{aligned}$$

$$\begin{aligned} |\vec{u}|^2 + |\vec{v}|^2 + 2(\vec{u} \cdot \vec{v}) &= 6 + 11 + 2((1, -1, 2) \cdot (1, -1, 3)) \\ &= 6 + 11 + 2(8) \\ &= 33 \end{aligned}$$

Observe that $(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = |\vec{u}|^2 + |\vec{v}|^2 + 2(\vec{u} \cdot \vec{v})$.

6. You need to show that the following is true for three vectors in two-space:

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

Let $\vec{u} = (x_1, y_1)$, $\vec{v} = (x_2, y_2)$, and $\vec{w} = (x_3, y_3)$:

$$\begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= (x_1, y_1) \cdot ((x_2, y_2) + (x_3, y_3)) \\ &= (x_1, y_1) \cdot (x_2 + x_3, y_2 + y_3) \\ &= x_1(x_2 + x_3) + y_1(y_2 + y_3) \\ &= x_1x_2 + x_1x_3 + y_1y_2 + y_1y_3 \\ &= x_1x_2 + y_1y_2 + x_1x_3 + y_1y_3 \\ &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \end{aligned}$$

7. $\vec{u} = (-2, 3, 0)$ and $\vec{v} = (1, 0, 1)$

$$\begin{array}{cccc}
 3 & & 0 & & -2 & & 3 \\
 \swarrow & & \searrow & & \swarrow & & \searrow \\
 0 & & 1 & & 1 & & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 (3)(1) - (0)(0) = 3 & & (0)(1) - (1)(-2) = 2 & & (-2)(0) - (1)(3) = -3 & &
 \end{array}$$

$$\vec{u} \times \vec{v} = (3, 2, -3)$$

You can confirm that the answer is correct by calculating the following dot products:

$$(-2, 3, 0) \cdot (3, 2, -3) = (-2)(3) + (3)(2) + (0)(-3) = 0$$

$$(1, 0, 1) \cdot (3, 2, -3) = (1)(3) + (0)(2) + (1)(-3) = 0$$

8. The cross product of two vectors is a vector that is perpendicular to both vectors:

$$\begin{array}{cccc}
 0 & & 0 & & 1 & & 0 \\
 \swarrow & & \searrow & & \swarrow & & \searrow \\
 1 & & 0 & & 0 & & 1 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 (0)(0) - (1)(0) = 0 & & (0)(0) - (0)(1) = 0 & & (1)(1) - (0)(0) = 1 & &
 \end{array}$$

$(0, 0, 1)$ is perpendicular to both $(1, 0, 0)$ and $(0, 1, 0)$.

You can confirm that the answer found is correct by calculating the following dot products:

$$(1, 0, 0) \cdot (0, 0, 1) = (1)(0) + (0)(0) + (0)(1) = 0$$

$$(0, 1, 0) \cdot (0, 0, 1) = (0)(0) + (1)(0) + (0)(1) = 0$$

Lesson 17

9. The area of the parallelogram is $|\vec{u} \times \vec{v}|$ where $\vec{u} = (1, 2, 4)$ and $\vec{v} = (-1, 3, -4)$. To find the area, compute $\vec{u} \times \vec{v}$.

$$\begin{array}{cccc}
 2 & 4 & 1 & 2 \\
 \swarrow & \searrow & \swarrow & \searrow \\
 3 & -4 & -1 & 3
 \end{array}$$

$$\begin{array}{c}
 \downarrow \\
 (2)(-4) - (3)(4) = -20
 \end{array}$$

$$\begin{array}{c}
 \downarrow \\
 (4)(-1) - (-4)(1) = 0
 \end{array}$$

$$\begin{array}{c}
 \downarrow \\
 (1)(3) - (-1)(2) = 5
 \end{array}$$

$$\vec{u} \times \vec{v} = (-20, 0, 5)$$

The area of the parallelogram:

$$|\vec{u} \times \vec{v}| = |(-20, 0, 5)| = \sqrt{(-20)^2 + 0^2 + 5^2} = \sqrt{400 + 0 + 25} = \sqrt{425} \approx 20.6.$$

10. The area of the triangle is half the area of the parallelogram determined by the vectors \vec{AB} and \vec{AC} .

$$\begin{aligned}
 \vec{AB} &= \vec{OB} - \vec{OA} \\
 &= (-3, 1, 0) - (-1, 0, 1) \\
 &= (-2, 1, -1)
 \end{aligned}$$

$$\begin{aligned}\vec{AC} &= \vec{OC} - \vec{OA} \\ &= (-2, -1, 2) - (-1, 0, 1) \\ &= (-1, -1, 1)\end{aligned}$$

Compute the cross product $\vec{AB} \times \vec{AC} = (-2, 1, -1) \times (-1, -1, 1)$

$$\begin{array}{ccccccc} 1 & & -1 & & -2 & & 1 \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ -1 & & 1 & & -1 & & -1 \end{array}$$

$$\begin{array}{c} \downarrow \\ (1)(1) - (-1)(-1) = 0 \end{array}$$

$$\begin{array}{c} \downarrow \\ (-1)(-1) - (1)(-2) = 3 \end{array}$$

$$\begin{array}{c} \downarrow \\ (-2)(-1) - (-1)(1) = 3 \end{array}$$

$$\vec{AB} \times \vec{AC} = (0, 3, 3)$$

The area of the parallelogram determined by \vec{AB} and \vec{AC} is $\sqrt{0^2 + 3^2 + 3^2} = \sqrt{0 + 9 + 9} = \sqrt{18} \approx 4.24$, and the area of the triangle is $\sqrt{18} \div 2 \approx 2.12$.

$$\begin{aligned}11. \quad V &= \left| (\vec{u} \times \vec{v}) \cdot \vec{w} \right| \\ &= \left| ((1, -1, 0) \times (1, 0, 3)) \cdot (0, -1, 1) \right| \\ &= \left| (-3, -3, 1) \cdot (0, -1, 1) \right| \\ &= \left| (-3)(0) + (-3)(-1) + (1)(1) \right| \\ &= |4| \\ &= 4\end{aligned}$$

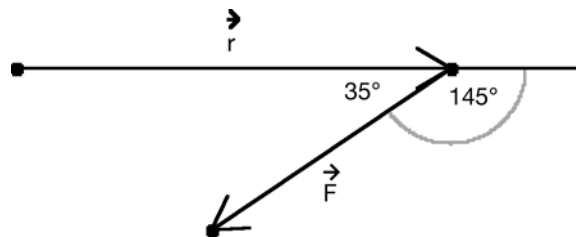
The volume of the parallelepiped is 4 cubic units.

$$\begin{aligned}
 12. \quad \text{proj}_{\vec{b}} \vec{a} &= \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \right) \vec{b} \\
 &= \frac{(4, -1, 4) \cdot (-2, 0, 4)}{(-2, 0, 4) \cdot (-2, 0, 4)} (-2, 0, 4) \\
 &= \frac{-8 + 0 + 16}{4 + 1 + 16} (-2, 0, 4) \\
 &= \frac{8}{21} (-2, 0, 4) \\
 &= \left(\frac{-16}{21}, 0, \frac{32}{21} \right)
 \end{aligned}$$

$$\begin{aligned}
 13. \quad W &= \vec{F} \cdot \vec{d} \\
 &= |\vec{F}| |\vec{d}| \cos \theta \\
 &= (40)(60) \cos(30) \\
 &\approx 2078.46
 \end{aligned}$$

The work done is approximately 2078.46 J.

14. a)



The torque vector, $\vec{\tau}$, is defined as

$$\begin{aligned}
 \vec{\tau} &= \vec{r} \times \vec{F} \\
 |\vec{\tau}| &= |\vec{r}| |\vec{F}| \sin \theta
 \end{aligned}$$

The wrench is 40 cm long, so $|\vec{r}| = 0.4$.

$$\theta = 180^\circ - 35^\circ$$

$$\theta = 145^\circ$$

$$\begin{aligned}
 |\vec{\tau}| &= |\vec{r}| |\vec{F}| \sin \theta \\
 &= (0.4)(70) \sin(145) \\
 &\approx 16.06
 \end{aligned}$$

The torque is approximately 16.06 Nm.

- b) The largest value of the sine function is 1, which happens when $\theta = 90^\circ$.

Calculate maximum torque:

$$\begin{aligned} |\vec{\tau}| &= |\vec{r}| |\vec{F}| \sin\theta \\ &= (0.4)(70)\sin(90) \\ &= 28 \end{aligned}$$

The maximum torque is 28 Nm. This is achieved when the force is applied at a 90° angle.

$$\begin{aligned} 15. \quad |\vec{\tau}| &= |\vec{r}| |\vec{F}| \sin\theta \\ &= (0.18)(100)\sin(75) \\ &\approx 17.39 \end{aligned}$$

The torque is approximately 17.39 Nm.

Lesson 18

16. The first line can be rewritten in slope-intercept form:

$$4x + 2y - 8 = 0$$

$$2y = -4x + 8$$

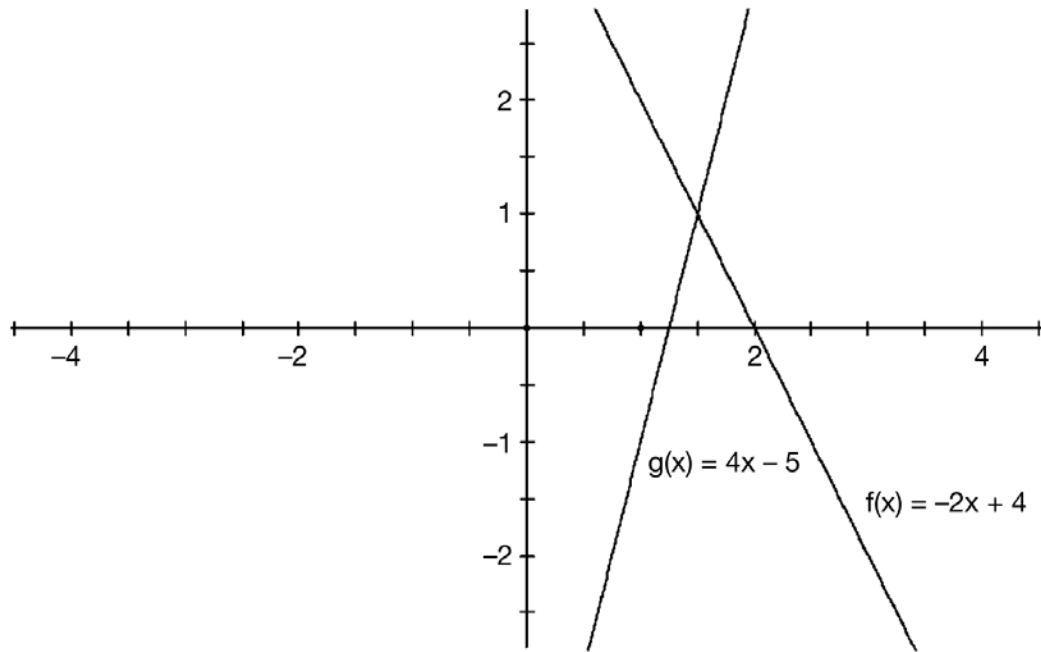
$$y = -2x + 4$$

The second equation can be rewritten in slope-intercept form:

$$-4x + y + 5 = 0$$

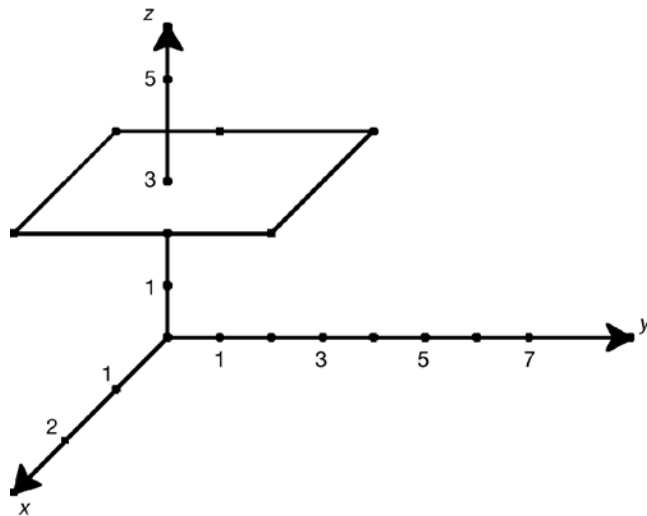
$$y = 4x - 5$$

The two equations can be graphed as follows:

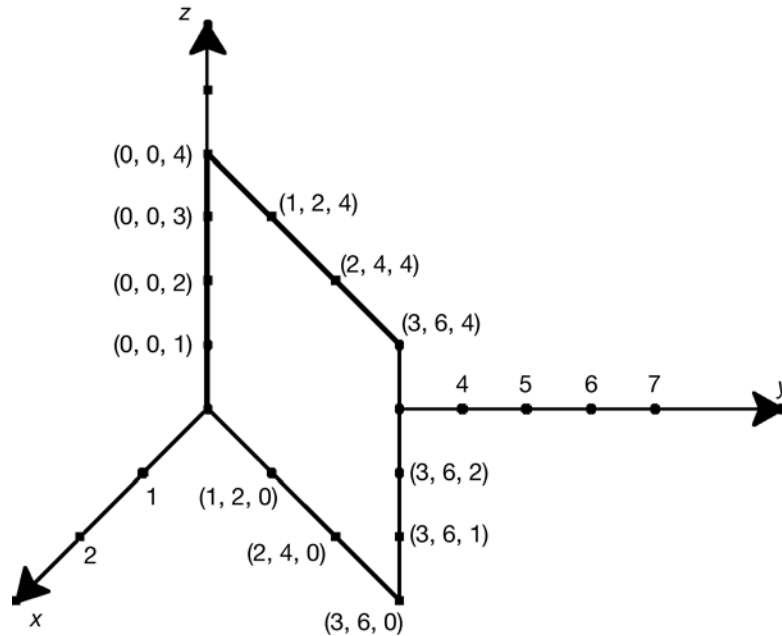


The intersection point appears to be approximately (1.5, 1).
Substituting these values of x and y into the original equations satisfies both equations.

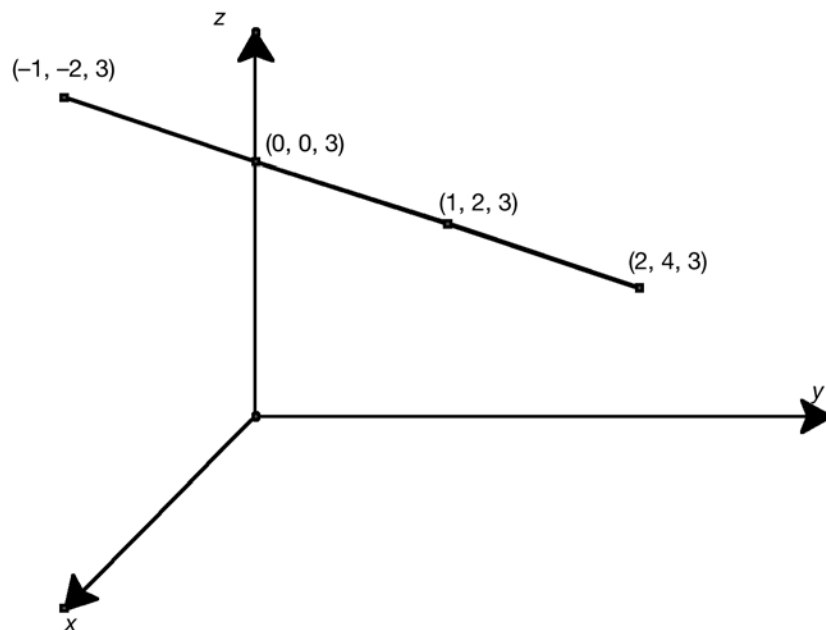
17. The plane $z = 3$ is parallel to the x - y plane, located 3 units along the z -axis in the positive direction.



The plane $y = 2x$ consists of all points where the y -coordinate is two times the x -coordinate, and the z -coordinate can have any value.



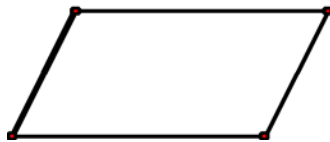
The intersection of these two planes is the line consisting of all points satisfying the relation $y = 2x$ that lie in the plane $z = 3$.





18. There are many possibilities:

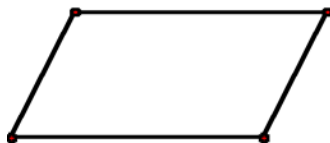
- All three planes are parallel. No intersection.



- All three planes are coincident. The intersection is a plane.

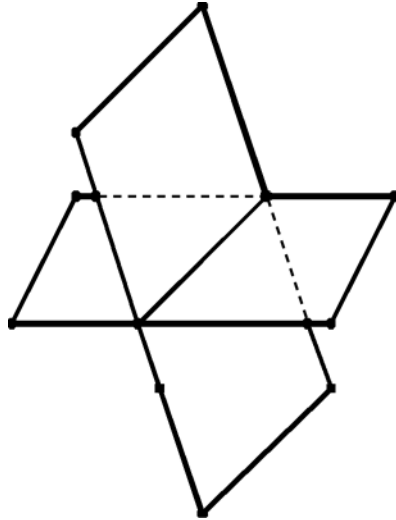


- Two planes are coincident and the other is parallel. The three planes do not intersect.

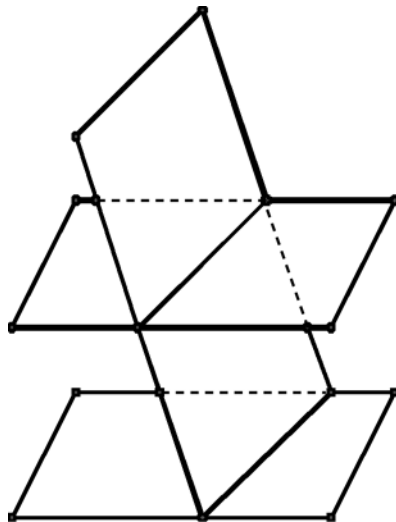




- Two planes are coincident and the other intersects the coincident planes in a line. The intersection is a line.

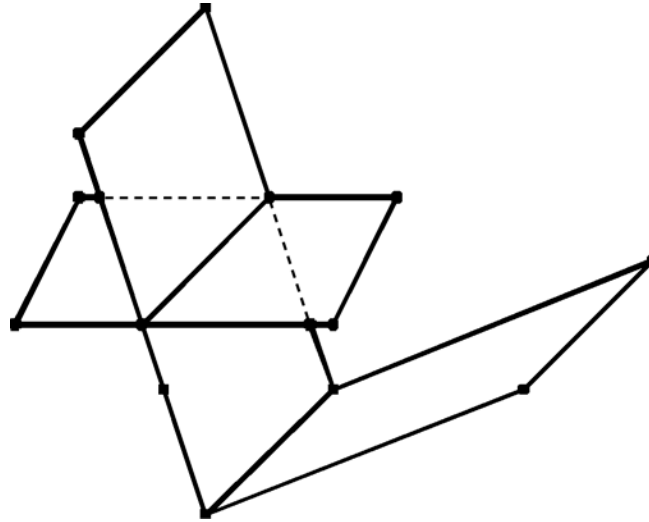


- Two planes are parallel and the other intersects both of them. The three planes do not intersect.

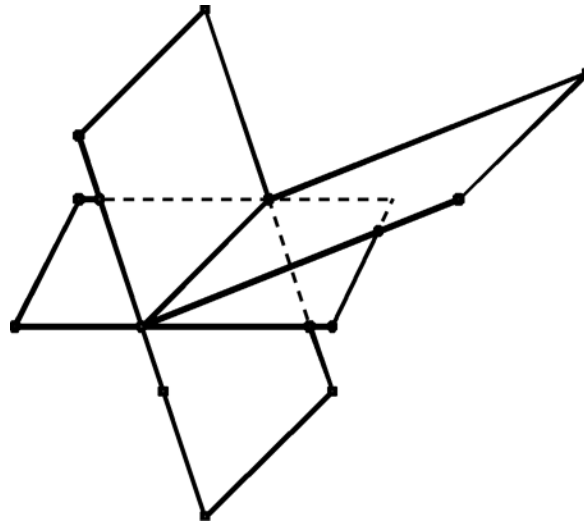




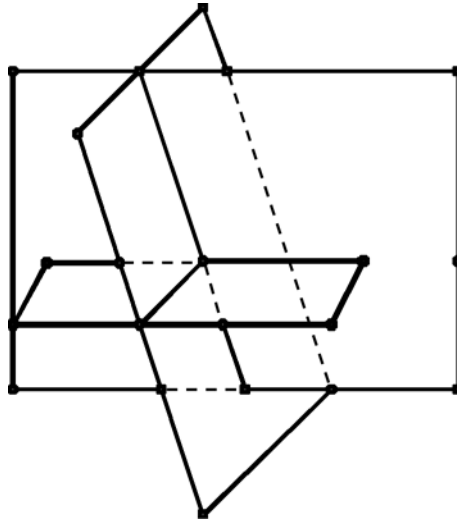
- None of the planes are parallel, and the three planes do not intersect.



- None of the planes are parallel, and the three planes intersect in a line.



- None of the planes are parallel, and the three planes intersect in a point.



Lesson 19

19. a) Find a direction vector for the line. An example of a direction vector is $\vec{AB} = \vec{OB} - \vec{OA} = (-3, 4) - (-1, 2) = (-2, 2)$. You also need one point on the line. You have been given two, so you can choose one arbitrarily. Suppose you choose A . An example of a vector equation for the line is $(x, y) = (-1, 2) + t(-2, 2)$, and an example of parametric equations for the line is

$$x = -1 - 2t$$

$$y = 2 + 2t$$

- b) You need one point on the line and a direction vector. You can find a direction vector if you know two points. You can find particular points on the line by substituting a particular value for x or y .

Substituting $x = 0$ gives $3y = 10$, so $(0, \frac{10}{3})$ is a point on the line. Substituting $y = 0$ gives $2x = 10$, so $(5, 0)$ is a point on the line.

Therefore, an example of a direction vector for the line is $(5, 0) - (0, \frac{10}{3}) = (5, -\frac{10}{3})$. (Note that $(0, \frac{10}{3}) - (5, 0)$ would be equally correct since that vector is also parallel to the line.)

A vector equation for the line is $(x, y) = (5, 0) + t(5, -\frac{10}{3})$ and parametric equations for the line are

$$x = 5 + 5t$$

$$y = 0 + (-\frac{10}{3})t$$

20. a) A vector equation for the line is simply $(x, y, z) = (1, -2, 2) + t(1, -1, 1)$ and the line can be represented with parametric equations as

$$x = 1 + t$$

$$y = -2 - t$$

$$z = 2 + t$$

- b) An example of a vector parallel to the x -axis is $(1, 0, 0)$, so you can use this as the direction vector. A vector equation for the line is $(x, y, z) = (-1, 0, 3) + t(1, 0, 0)$ and parametric equations are

$$x = -1 + t$$

$$y = 0$$

$$z = 3$$

- c) A direction vector is $(-1, 2, 3) - (1, 3, 4) = (-2, -1, -1)$. Thus, a vector equation for the line is $(x, y, z) = (-1, 2, 3) + t(-2, -1, -1)$ and parametric equations are

$$x = -1 - 2t$$

$$y = 2 - t$$

$$z = 3 - t$$

21. To get a scalar equation, you need a normal vector. You can find a normal vector by taking the cross product of two vectors in the plane. Here are two examples of vectors in the plane:

$$\vec{AB} = (0, -2, 2) - (3, 2, 5) = (-3, -4, -3)$$

$$\vec{AC} = (1, 3, 1) - (3, 2, 5) = (-2, 1, -4)$$

Compute the cross product $(-3, -4, -3) \times (-2, 1, -4)$:

$$\begin{array}{cccc}
 -4 & & -3 & & -3 & & -4 \\
 & \nearrow & & \nearrow & & \nearrow & \\
 & & -4 & & -2 & & \\
 & \searrow & & \searrow & & \searrow & \\
 1 & & -4 & & -2 & & 1
 \end{array}$$

\downarrow

$$(-4)(-4) - (1)(-3) = 19$$

\downarrow

$$(-3)(-2) - (-4)(-3) = -6$$

\downarrow

$$(-3)(1) - (-2)(-4) = -11$$

$$(-3, -4, -3) \times (-2, 1, -4) = (19, -6, -11)$$

A scalar equation for the plane will have the following form:

$$19x - 6y - 11z + d = 0$$

Substitute one of the given points to find the value of d .

For example, using $(1, 3, 1)$ gives the following:

$$19(1) - 6(3) - 11(1) + d = 0$$

$$19 - 18 - 11 + d = 0$$

$$d = 10$$

A scalar equation of the plane is $19x - 6y - 11z + 10 = 0$.

Since you have two direction vectors and at least one point on the plane, it's easy to write a vector equation for the plane:

$$(x, y, z) = (3, 2, 5) + s(-3, -4, -3) + t(-2, 1, -4)$$

These are the parametric equations:

$$x = 3 - 3s - 2t$$

$$y = 2 - 4s + t$$

$$z = 5 - 3s - 4t$$

22. If two planes are parallel, they have the same normal vector. You can find a normal vector for the given plane by taking the cross product of the two given direction vectors. Use the usual process to find the cross product:

$$(1, 2, -1) \times (0, -2, 1) = (0, -1, -2)$$

The given plane, and any plane parallel to it, will have normal vector $(0, -1, -2)$ and scalar equation of the form $0x - y - 2z + d = 0$.

Since the plane must contain the point $(1, 0, 2)$, substitute these coordinates to find the value of d :

$$(0)(1) - (1)(0) - (2)(2) + d = 0$$

$$0 - 0 - 4 + d = 0$$

$$d = 4$$

A scalar equation of the plane is $-y - 2z + 4 = 0$.

Lesson 20

23. a) $\vec{l}_1 = (1, 3, 2) + t(1, -2, 4)$ and $\vec{l}_2 = (2, 1, 8) + s(1, -2, 6)$

$\vec{d}_1 = (1, -2, 4)$ is a direction vector of \vec{l}_1 and $\vec{d}_2 = (1, -2, 6)$ is a direction vector of \vec{l}_2 .

The direction vectors of the two lines are not collinear since the ratios of the components are not equal. Therefore, the two lines intersect in a point.

Write the two lines in parametric form:

$$x = 1 + t$$

$$y = 3 - 2t$$

$$z = 2 + 4t$$

$$x = 2 + s$$

$$y = 1 - 2s$$

$$z = 8 + 6s$$

A point on both lines must satisfy the equations of the two lines. Find values of s and t that satisfy the following linear system:

$$1 + t = 2 + s \quad [1]$$

$$3 - 2t = 1 - 2s \quad [2]$$

$$2 + 4t = 8 + 6s \quad [3]$$

$$[1] \rightarrow t - s = 1 \quad [4]$$

$$[3] \rightarrow 4t - 6s = 6 \quad [5]$$

$$[4] \times 4 \quad 4t - 4s = 4$$

$$[5] \quad 4t - 6s = 6$$

Subtract:

$$2s = -2$$

$$s = -1$$

Substitute $s = -1$ into [4] to find t :

$$t - s = 1$$

$$t = 1 + (-1)$$

$$t = 0$$

Substitute $t = 0$ and $s = -1$ into both sides of [2] to check for consistency:

LS:

$$3 - 2t = 3$$

RS:

$$\begin{aligned} 1 - 2s &= 1 - 2(-1) \\ &= 3 \end{aligned}$$

The two values are the same, so the two lines intersect.

To find the point of intersection, substitute $t = 0$ or $s = -1$ into the equations of either lines. Use $t = 0$.

$$x = 1, y = 3, \text{ and } z = 2$$

The two lines intersect at $(1, 3, 2)$.

b) $\vec{l}_1 = (2, 1, 0) + t(2, 4, -10)$ and $\vec{l}_2 = (11, 2, -1) + s(-8, -16, 40)$

$\vec{d}_1 = (2, 4, -10)$ is a direction vector of \vec{l}_1 and $\vec{d}_2 = (-8, -16, 40)$ is a direction vector of \vec{l}_2 . The two vectors are collinear since $\vec{d}_2 = -4\vec{d}_1$.

If the two lines have a point in common, they are coincident; otherwise, they are parallel.

The point $(2, 1, 0)$ is on \vec{l}_1 . Check to see if it is also on \vec{l}_2 :

$$(2, 1, 0) = (11, 2, -1) + s(-8, -16, 40)$$

$$2 = 11 - 8s \rightarrow 8s = 9, s = \frac{9}{8}$$

$$1 = 2 - 16s \rightarrow 16s = 1, s = \frac{1}{16}$$

$$0 = -1 + 40s$$

$(2, 1, 0)$ is not on \vec{l}_2 . The two lines are parallel.

24. Two linear equations to solve:

$$2x - 3y - z + 1 = 0$$

$$3x - 2y + 3z - 4 = 0$$

Eliminate the variable z from these equations by multiplying the first equation by 3 and then adding:

$$6x - 9y - 3z + 3 = 0$$

$$3x - 2y + 3z - 4 = 0$$

$$\hline 9x - 11y - 1 = 0$$

Choose one of the variables x or y to be the parameter. For example, let $y = t$:

$$9x - 11t - 1 = 0$$

Solve for x by rearranging:

$$9x = 11t + 1$$

$$x = \frac{11}{9}t + \frac{1}{9}$$

x and y are expressed in terms of the parameter t . Substitute $x = \frac{11}{9}t + \frac{1}{9}$ and $y = t$ into one of the equations of the given planes, such as the equation $2x - 3y - z + 1 = 0$:

$$2\left(\frac{11}{9}t + \frac{1}{9}\right) - 3t - z + 1 = 0$$

$$\frac{22}{9}t + \frac{2}{9} - 3t - z + 1 = 0$$

$$\frac{22}{9}t + \frac{2}{9} - \frac{27}{9}t - z + \frac{9}{9} = 0$$

$$\frac{22}{9}t - \frac{27}{9}t - z = -\frac{2}{9} - \frac{9}{9}$$

$$\frac{-5}{9}t - z = \frac{-11}{9}$$

$$-z = \frac{5}{9}t - \frac{11}{9}$$

$$z = \frac{-5}{9}t + \frac{11}{9}$$

All three variables (x , y , and z) are expressed in terms of the parameter t . Here are parametric equations of the line:

$$x = \frac{11}{9}t + \frac{1}{9}$$

$$y = t$$

$$z = \frac{-5}{9}t + \frac{11}{9}$$

25. Find the intersection of the two planes. Eliminate the variable y by multiplying the first equation by 2 and then adding:

$$2x + 2y + 6z - 20 = 0$$

$$6x - 2y + z - 10 = 0$$

$$\hline 8x + 7z - 30 = 0$$

Let $z = t$:

$$8x + 7t - 30 = 0$$

$$8x = -7t + 30$$

$$x = \frac{-7}{8}t + \frac{30}{8}$$

Substitute $z = t$ and $x = \frac{-7}{8}t + \frac{30}{8}$ into the equation of one of the planes, say the first plane $x + y + 3z - 10 = 0$:

$$\left(\frac{-7}{8}t + \frac{30}{8}\right) + y + 3t - 10 = 0$$

$$y = \frac{7}{8}t - 3t - \frac{30}{8} + 10 = \frac{7}{8}t - \frac{24}{8}t - \frac{30}{8} + \frac{80}{8}$$

$$y = \frac{-17}{8}t + \frac{50}{8}$$

x , y , and z are expressed in terms of the parameter t ,
parametric equations of the intersection of the two planes:

$$x = \frac{-7}{8}t + \frac{30}{8}$$

$$y = \frac{-17}{8}t + \frac{50}{8}$$

$$z = 1t + 0$$

This means that a direction vector for the line is $(\frac{-7}{8}, \frac{-17}{8}, 1)$.
Here is the equation for the line:

$$(x, y, z) = (3, -1, 2) + t(\frac{-7}{8}, \frac{-17}{8}, 1)$$

26. Use the elimination method to solve for the intersection.
Label the equations [1], [2], and [3]:

$$3x + z + 11 = 0 \quad [1]$$

$$2x + y + z + 4 = 0 \quad [2]$$

$$x + y + z - 3 = 0 \quad [3]$$

Eliminate variable z :

$$3x + z + 11 = 0 \quad [1]$$

$$2x + y + z + 4 = 0 \quad [2]$$

Subtract:

$$x - y + 7 = 0 \quad [4]$$

$$2x + y + z + 4 = 0 \quad [2]$$

$$x + y + z - 3 = 0 \quad [3]$$

Subtract:

$$x + 7 = 0$$

$$x = -7$$

Substitute $x = -7$ into [1]:

$$3x + z + 11 = 0$$

$$3(-7) + z + 11 = 0$$

$$-21 + z + 11 = 0$$

$$z = 10$$

Substitute $x = -7$ and $z = 10$ into [2]:

$$2x + y + z + 4 = 0 \quad [2]$$

$$2(-7) + y + 10 + 4 = 0$$

$$-14 + y + 14 = 0$$

$$y = 0$$

The system of linear equations has a solution $x = -7$, $y = 0$, and $z = 10$.

Geometrically, the three planes represented by the linear equations intersect at the point $(-7, 0, 10)$.

You can confirm that the three planes intersect at a point by calculating $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)$ and asserting that $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$.

27.

$$x - 2y - 2z - 6 = 0 \quad [1]$$

$$2x - 5y + 3z + 10 = 0 \quad [2]$$

$$3x - 7y + z + 9 = 0 \quad [3]$$

$$[1] \times 2 \quad 2x - 4y - 4z - 12 = 0$$

$$[2] \quad 2x - 5y + 3z + 10 = 0$$

$$\text{Subtract: } y - 7z - 22 = 0 \quad [4]$$

$$[1] \times 3 \quad 3x - 6y - 6z - 18 = 0$$

$$[3] \quad 3x - 7y + z + 9 = 0$$

$$\text{Subtract: } y - 7z - 27 = 0 \quad [5]$$

$$y - 7z - 22 = 0 \quad [4]$$

$$y - 7z - 27 = 0 \quad [5]$$

Subtract:

$$5 = 0$$

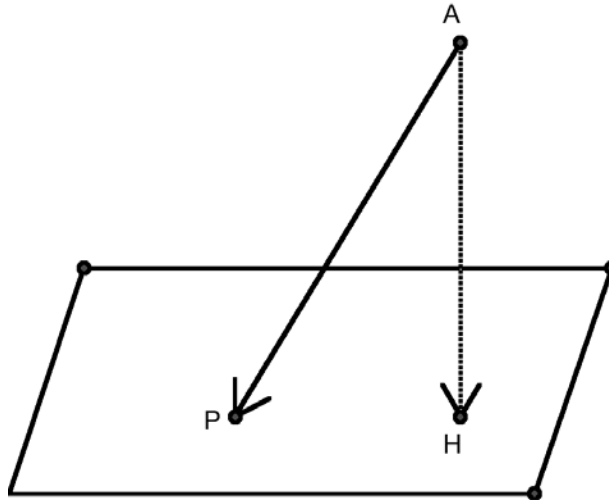
This is obviously wrong.

You can conclude that the system has no solution.

28. Start with any point on the plane:

$$P\left(0, 0, \frac{2}{5}\right)$$

$$\vec{AP} = \left(2, -1, -\frac{8}{5}\right)$$



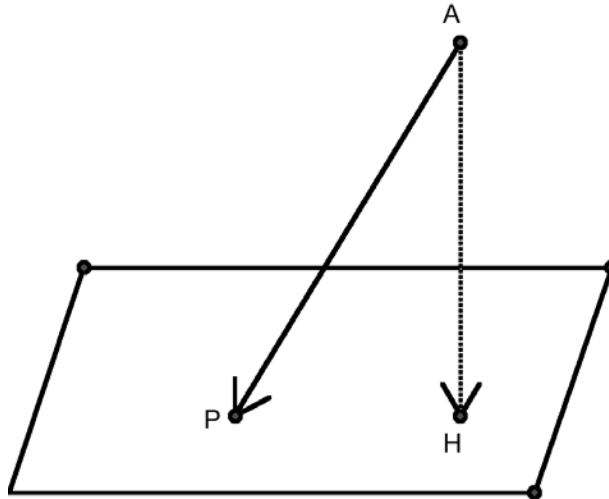
$$\begin{aligned} |\vec{AH}| &= \left| \text{proj}_{\vec{n}}(\vec{PA}) \right| \\ &= \left| \frac{\vec{PA} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right| \\ &= \frac{|\vec{PA} \cdot \vec{n}|}{|\vec{n}|} \\ &= \left| \frac{\left(2, -1, -\frac{8}{5}\right) \cdot (3, -2, 5)}{(3, -2, 5) \cdot (3, -2, 5)} \right| \sqrt{(3)^2 + (-2)^2 + (5)^2} \\ &= \left| \frac{6 + 2 - 8}{(9 + 4 + 25)} \right| \sqrt{9 + 4 + 25} \\ &= \frac{0}{38} \sqrt{38} \\ &= 0 \end{aligned}$$

This means that point A is on the plane.

29. Start with any point on the plane:

$$P(4, 0, 0)$$

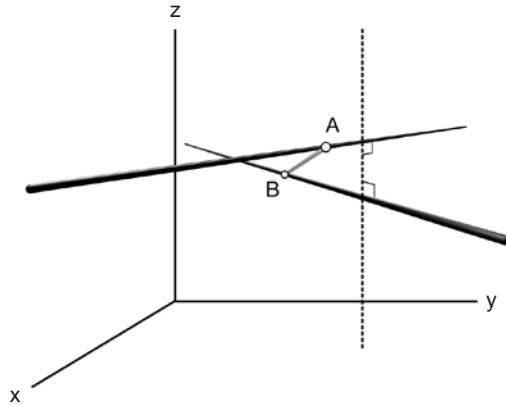
$$\vec{AP} = (3, 2, -3)$$



$$\begin{aligned} |\vec{AH}| &= \left| \text{proj}_{\vec{n}}(\vec{PA}) \right| \\ &= \left| \frac{\vec{PA} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right| \\ &= \left| \frac{\vec{PA} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \right| |\vec{n}| \\ &= \left| \frac{(3, 2, -3) \cdot (2, -3, 6)}{(2, -3, 6) \cdot (2, -3, 6)} \right| \sqrt{(2)^2 + (-3)^2 + (6)^2} \\ &= \left| \frac{6 + (-6) + (-18)}{(4 + 9 + 36)} \right| \sqrt{4 + 9 + 36} \\ &= \left| \frac{-18}{49} \right| \sqrt{49} \\ &= \frac{18}{7} \end{aligned}$$

The distance from point A to the plane is $\frac{18}{7}$.

30. The two skew lines $\vec{l}_1 = (6, -4, 0) + t(0, 1, -1)$ and $\vec{l}_2 = (0, 5, 3) + s(2, 0, -1)$ are illustrated in the following diagram. The vertical dashed line is a common perpendicular to the two skew lines:



The projection of \vec{AB} onto the normal is the distance between the two skew lines.

$A(-6, -4, 0)$ is a point on \vec{l}_1 and $B(0, 5, 3)$ is a point on \vec{l}_2 .

$$\vec{AB} = \vec{OB} - \vec{OA} = (6, 9, 3)$$

$\vec{d}_1 = (0, 1, -1)$ is a direction vector of \vec{l}_1 and $\vec{d}_2 = (2, 0, -1)$ is a direction vector of \vec{l}_2 .

$\vec{d}_1 \times \vec{d}_2$ is perpendicular to both lines. Using the usual procedure you find:

$$\vec{n} = \vec{d}_1 \times \vec{d}_2 = (-1, -2, -2)$$

The distance between \vec{l}_1 and \vec{l}_2 is $\left| \text{proj}_{\vec{n}}(\vec{AB}) \right|$.

$$\begin{aligned} & \left| \text{proj}_{\vec{n}}(\vec{AB}) \right| \\ &= \left| \frac{\vec{AB} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right| \\ &= \frac{\left| \vec{AB} \cdot \vec{n} \right|}{\left| \vec{n} \cdot \vec{n} \right|} \left| \vec{n} \right| \\ &= \frac{\left| (6, 9, 3) \cdot (-1, -2, -2) \right|}{\left| (-1, -2, -2) \cdot (-1, -2, -2) \right|} \sqrt{(-1)^2 + (-2)^2 + (-2)^2} \\ &= \frac{\left| (6)(-1) + (9)(-2) + (3)(-2) \right|}{(1 + 4 + 4)} \sqrt{1 + 4 + 4} \\ &= \left| \frac{-30}{9} \right| \sqrt{9} \\ &= 10 \end{aligned}$$

The distance between the two skew lines is 10.