

Duration: 50 minutes.

Total marks:30

NAME (in ink):

STUDENT NO (in ink):

1. a) [2] Let the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the reflection in the x-axis. Give the matrix of the transformation.

ANS: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

b)[2] Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation. Suppose $T(u) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $T(v) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Calculate $T(3u - 2v)$.

ANS: $T(3u - 2v) = 3T(u) - 2T(v) = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

c)[2] Let the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ rotate points about origin through an angle of $\pi/3$ radians. What is the standard matrix of the transformation?

ANS: Under the transformation ,

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\pi/3) \\ \sin(\pi/3) \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(\pi/3) \\ \cos(\pi/3) \end{bmatrix}$$

So the standard matrix is

$$\begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

d)[2] [2] Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear transformation s.t. $T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$, $T \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$.

Determine $T(\vec{x})$ where $\vec{x} = \begin{bmatrix} 5 \\ -1 \\ 2 \\ -4 \end{bmatrix}$

ANS: $\left[\begin{array}{ccc|c} 1 & -1 & 5 & \\ 1 & 1 & -1 & \\ 1 & 0 & 2 & \\ 1 & 2 & -4 & \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & \\ 0 & 1 & -3 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{array} \right], x = \begin{bmatrix} 5 \\ -1 \\ 2 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. So, $T(x) = 2T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - 3T \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \\ -5 \end{bmatrix}$

2. [11] Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation such that

$$T(x_1, x_2, x_3) = T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 + 2x_2 - x_3 \\ -3x_1 + x_2 + 5x_3 \\ 6x_1 + 5x_2 + x_3 \end{bmatrix}$$

a) [2] Find the standard matrix of T .

ANS: $T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 6 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}$

The standard matrix of T is $A = \begin{bmatrix} 3 & 2 & -1 \\ -3 & 1 & 5 \\ 6 & 5 & 1 \end{bmatrix}$

b) [2] Find the image of the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ under T .

ANS: $\begin{bmatrix} 3 & 2 & -1 \\ -3 & 1 & 5 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -2 \\ 29 \end{bmatrix}$

c) [3] Is the vector $\begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$ in the range of T . Explain.

ANS: We need to find a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. $T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ -3 & 1 & 5 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 & -1 \\ -3 & 1 & 5 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since the above system is consistent, the vector $\begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$ is in the range of T .

d) [2] Is T “onto”? Explain.

ANS: $A = \begin{bmatrix} 3 & 2 & -1 \\ -3 & 1 & 5 \\ 6 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since each row has a pivot position, columns of A span \mathfrak{R}^3 . Therefore, T is “onto”.

e) [2] Is “one-to-one”? Explain.

ANS: Since each column of RREF of A is a pivot column, columns of A are linearly independent. Therefore T “one-to-one”.

3. [14] Let $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

(a) [3] Write the characteristic equation of A .

ANS: $\det(A - \lambda I) = \begin{vmatrix} 4 - \lambda & 0 & -2 \\ 2 & 5 - \lambda & 4 \\ 0 & 0 & 5 - \lambda \end{vmatrix}$.

Expanding by the last row,

$$\det(A - \lambda I) = (5 - \lambda)(5 - \lambda)(4 - \lambda)$$

So the characteristic equation of A is $(5 - \lambda)(5 - \lambda)(4 - \lambda) = 0$

$$\text{Or, } \lambda^3 - 14\lambda^2 + 65\lambda - 100 = 0$$

NOTE: any of the above equations is alright.

(b) [2] Find the eigenvalues of the matrix A .

ANS: Eigenvalues of the matrix A are 5, 5, 4

(c) [6] For each eigenvalue, find a basis for the corresponding eigenspace.

ANS: Basis corresponding to eigenspace of $\lambda = 4$:

$$\text{Solve } (A - 4I)X = \vec{0}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{So, } X = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} x_2; x_2 \in \mathfrak{R}.$$

A basis corresponding to eigenspace of $\lambda = 4$ is $\begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$.

Basis corresponding to eigenspace of $\lambda = 5$:

$$\text{Solve } (A - 5I)X = \vec{0}$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{So, } X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} x_3; x_2, x_3 \in R.$$

A basis corresponding to eigenspace of $\lambda = 5$ is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

(d) [3] Find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

ANS:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}, P = \begin{bmatrix} -1/2 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$