Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322B - Fall 2016
Midterm I
Professor: C.Rada
Time limit: 80 minutes. Closed books.

Name: HA-HA-HA ID Number: -SOL-

Cellular phones, unauthorized electronic devices or course notes are not allowed during this exam. Phones and devices must be turned off and put away in your bag. Do not keep them in your possession, such as in your pockets. If caught with such a device or document, the following may occur: you will be asked to leave immediately the exam and academic fraud allegations will be filed which may result in you obtaining a 0 (zero) for the exam. By signing below, you acknowledge that you have ensured that you are complying with the above statement.

(Signature): ______________________

Instructions

- **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- The exam has 8 pages. Read each question carefully before answering.
- Questions 1 to 3 are multiple choice. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 4 to 6 are long answer questions. Questions 4 and 6 are worth 6 marks each, and question 5 is worth 7 marks, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstaple the test. Good luck!

Answers to multiple choice Qs

1	2	3
C	A	F

Grid below is used for grading
(do not write in this grid)

MCQ	4	5	6	Total
/6	/6	/7	/6	/25

easy .....

$\begin{cases} f_x = 2x \\ f_y = -6y \end{cases} \Rightarrow \vec{\nabla} f(x,y) = \langle 2x, -6y \rangle \Rightarrow \nabla f(3,2) = \langle 6, -12 \rangle$

1. For all possible unit vectors $\vec{u}$, what is the largest possible value of $D_{\vec{u}}f(3,2)$, where $f(x,y) = x^2 - 3y^2$?

So:

$|\nabla f(3,2)| = \sqrt{6^2 + (-12)^2} = \sqrt{6^2 + 6^2 \cdot 2^2} = 6\sqrt{1+4} = 6\sqrt{5}$

A. $8\sqrt{7}$

B. $7\sqrt{6}$

C. $6\sqrt{5}$

D. $5\sqrt{4}$

E. $4\sqrt{3}$

F. $3\sqrt{2}$

2. Which of the following statements is true concerning the critical points of the function $f(x,y) = x^4 - 4x^3 + 4x^2 + y^2$?

A. $(0,0)$ is a local minimum, $(1,0)$ is a saddle, $(2,0)$ is a local minimum

B. f has no critical points

C. $(0,0)$ is a local maximum, $(1,0)$ is a saddle, $(2,0)$ is a local maximum

D. $(0,0)$ is a local minimum, $(1,0)$ is a local maximum, $(2,0)$ is a local minimum

E. $(0,0)$ is a local maximum, $(1,0)$ is a local minimum, $(2,0)$ is a local maximum

F. None of the above

$\begin{cases} f_x = 4x^3 - 12x^2 + 8x = 0 \\ f_y = 2y = 0 \end{cases}$

$\begin{cases} 4x(x-1)(x-2) = 0 \\ y = 0 \end{cases}$

So: $(0,0)$, $(1,0)$, $(2,0)$ are C.P.

NOTE: $f_{xx} = 12x^2 - 24x + 8$; $f_{yy} = 2$; $f_{xy} = 0$

$\begin{cases} \text{FOR } (0,0) \Rightarrow D(0,0) = 8 \cdot 2 - 0^2 = 16 > 0 ; \; f_{xx}(0,0) = 8 > 0 \\ \text{FOR } (1,0) \Rightarrow D(1,0) = (-4) 2 - 0^2 = -8 < 0 ; \\ \text{FOR } (2,0) \to D(2,0) = 8 \cdot 2 - 0^2 > 0 ; \; f_{xx}(2,0) = 8 > 0 \end{cases}$

3. For the composite function $z(u,v) = f(x(u,v), y(u,v))$, which of the following correspond to the chain rule for $\dfrac{\partial z}{\partial v}$?

A. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

B. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial v} + \dfrac{\partial z}{\partial v}$

C. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

D. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v}$

E. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

F. $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v}$

4. Compute the tangent plane to the graph of the function $z = f(x, y) = x \cos y + y \sin x$ at the point $(x, y, z) = (\pi/2, \pi/2, \pi/2)$.

$\frac{\partial f}{\partial x}(x,y) = \cos y + y \cdot \cos x$; ~~AND~~ $\frac{\partial f}{\partial y}(x,y) = x(-\sin y) + \sin$

$\frac{\partial f}{\partial x}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 0$ and $\frac{\partial f}{\partial y}\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -\frac{\pi}{2} + 1$

$$\boxed{z = \frac{\pi}{2} + 0 \cdot \left(x - \frac{\pi}{2}\right) + \left(-\frac{\pi}{2} + 1\right) \cdot \left(y - \frac{\pi}{2}\right)}$$

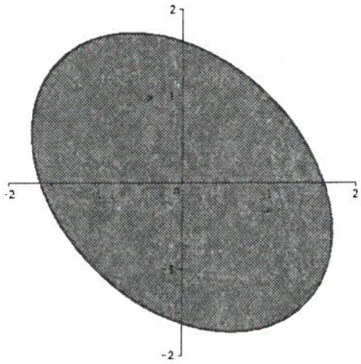
Step 1°) $f_x = 2x$, $f_y = 2y$. Solve $\begin{cases} 2x=0 \\ 2y=0 \end{cases} \to \begin{cases} x=0 \\ y=0 \end{cases} \to (0,0) \overset{\circ}{\in} R = \mathcal{D}$!!

is a C. POINT.

5. Find the global extrema of the function $f(x,y) = x^2 + y^2$ over the elliptical domain $D = \{ (x,y) \in \mathbb{R}^2 \,|\, 3x^2 + 2xy + 3y^2 \leq 8 \}$ sketched below. For part of your solution, you must use the method of Lagrange multipliers.

$\boxed{f(0,0) = 0^2 + 0^2 = 0}$

step 2°) Set up:



$\begin{cases} \nabla f = \lambda \cdot \nabla g \\ g(x,y) = 8 \\ g(x,y) = 3x^2 + 2xy + 3y^2 \end{cases}$

$\begin{cases} (2x, 2y) = \lambda (6x+2y, 2x+6y) \\ 3x^2 + 2xy + 3y^2 = 8 \end{cases}$

so: $\begin{cases} x = \lambda(3x+y) & \text{EQ 1} \\ y = \lambda(x+3y) & \text{EQ 2} \\ 3x^2+2xy+3y^2 = 8 & \text{EQ 3} \end{cases}$

There is only One $\boxed{\text{idea}}$: ADD $\boxed{1, 2}$

$(x+y) = 4\lambda x + 4\lambda y \Rightarrow (x+y) = (x+y)(4\lambda)$

or $\boxed{(x+y)(1-4\lambda) = 0}$

CASE 1°) $x+y=0$ or $\boxed{x=-y}$

EQ 1 gives: $x = \lambda(3x-x) \Rightarrow x = 2\lambda x \Rightarrow x(1-2\lambda) = 0$.

case $\boxed{1°.1}$ $x=0$; so $y = -x = -0 \to (x,y) = (0,0)$, but:

$3 \cdot 0^2 + 2 \cdot 0 \cdot 0 + 3 \cdot 0^2 = 0 \neq 8$ (see EQ 3). So: NOT good!

$\boxed{\text{case } 1°.2}$ $x \neq 0$; so $1 - 2\lambda = 0 \to \boxed{\lambda = \frac{1}{2}}$

EQ 1 gives: $x = \frac{3}{2}x + \frac{y}{2} \Rightarrow -\frac{x}{2} = \frac{y}{2} \Rightarrow -x = y$

Then EQ 3 gives: $3x^2 + 2x(-x) + 3x^2 = 8 \to 4x^2 = 8 \Rightarrow x^2 = 2$

$\Rightarrow x = \pm\sqrt{2}$. So: $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$.

In EACH case: $\boxed{f(\sqrt{2}, -\sqrt{2}) = 4 = f(-\sqrt{2}, \sqrt{2})}$

(if you are ready to see step 2°, case 2°

P.T.O.

case 2°) $x+y\neq 0$. So $1-4\lambda = 0$

or $\lambda = \frac{1}{4}$

(Extra page)

Eq 1 gives: $x = \frac{3}{4}x + \frac{y}{4} \Rightarrow \boxed{x=y}$

Then Eq 3 gives: $3x^2 + 2x^2 + 3x^2 = 8 \to x^2 = 1 \to$

$x = \pm 1$. So: $(1,1)$; $(-1,-1)$.

In each case $\boxed{f(1,1) = f(-1,-1) = 2}$

Step 3°) Decide:

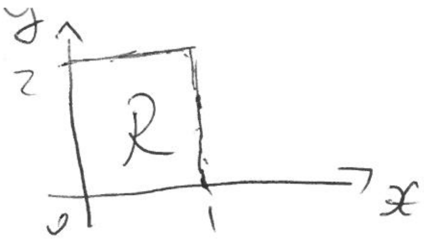
GLOBAL MIN at $(0,0)$; value: 0

GLOBAL MAX at $(\sqrt{2},-\sqrt{2})$ or $(-\sqrt{2},\sqrt{2})$; value 4

WAS it hard??! NO!

f, g were symmetric .... that WAS the idea ....

Algebraic manipulations MUST be mastered by all!



6. Compute the double integral of the function $f(x,y) = 12\,x^2y^3$ over the rectangle

$$R = \{\,(x,y) \in \mathbb{R}^2 \,|\, 0 \le x \le 1, 0 \le y \le 2\}$$

$$\iint_R 12\,x^2y^3\,dA = 12 \cdot \int_0^1 \left(\int_0^2 x^2y^3\,dy \right) dx =$$

$$= 12 \cdot \int_0^1 \left[x^2 \cdot \frac{y^4}{4} \Big|_0^2 \right] dx = 12 \int_0^1 x^2 \cdot \frac{2^4}{4}\,dx$$

$$= 12 \cdot \frac{2^4}{4} \cdot \left[\frac{x^3}{3} \Big|_0^1 \right] = 12 \cdot \frac{16}{4} \left[\frac{1}{3} - 0 \right]$$

$$= 12 \cdot 4 \cdot \frac{1}{3} = \boxed{16}$$