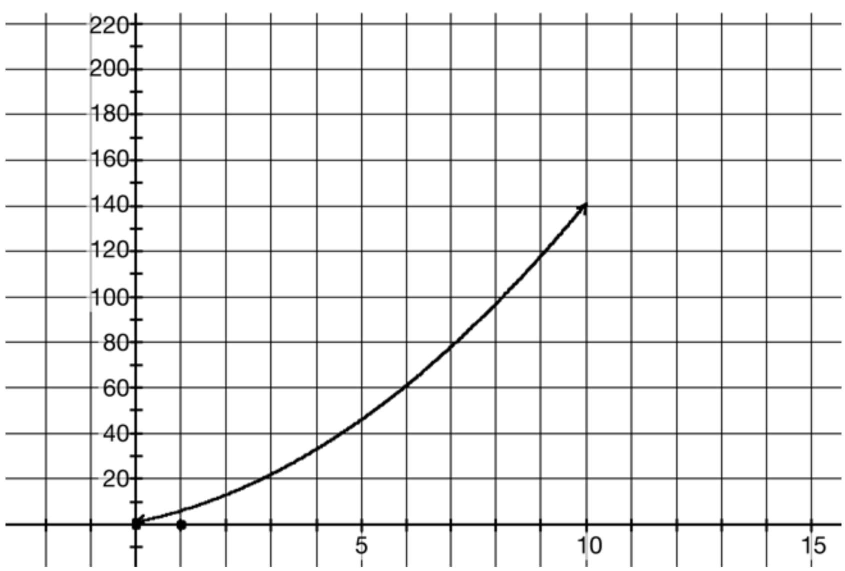
Unit 1

Lesson 1

1. a) Table of values:

t	*N*
0	1
1	6
2	$(2)^2 + 4(2) + 1 = 13$
3	22
5	46
6	61
8	97
10	141

Draw a smooth curve that passes through the ordered pairs to get the graph:



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b) Average rate of change between 1 and 10 seconds:

Average Rate of Change = $\dfrac{f(b)-f(a)}{b-a}$

$N(1) = 6$ and $N(10) = 141$

The average rate of change of the number of insects between 1 and 10 seconds is the slope of the secant connecting $A(1, 6)$ and $B(10, 141)$.

The average rate of change between 10 and 1:

$$\frac{N(10)-N(1)}{10-1}$$
$$=\frac{141-6}{10-1}$$
$$=\frac{135}{9}$$
$= 15$ insects/sec

c) Approximate the rate of change at $t = 5$ by calculating the slope of the secant between $t = 4.5$ and $t = 5$:

$$\frac{N(5)-N(4.5)}{5-4.5}=\frac{46-39.25}{0.5}$$
$$= 13.5$$

Note that your answer may be different if you choose a different interval.

2. a) Table of values:

t	***h***
0	0
1	25.1
2	40.4
3	45.9
4	41.6
5	27.5
6	3.6
7	–30.1

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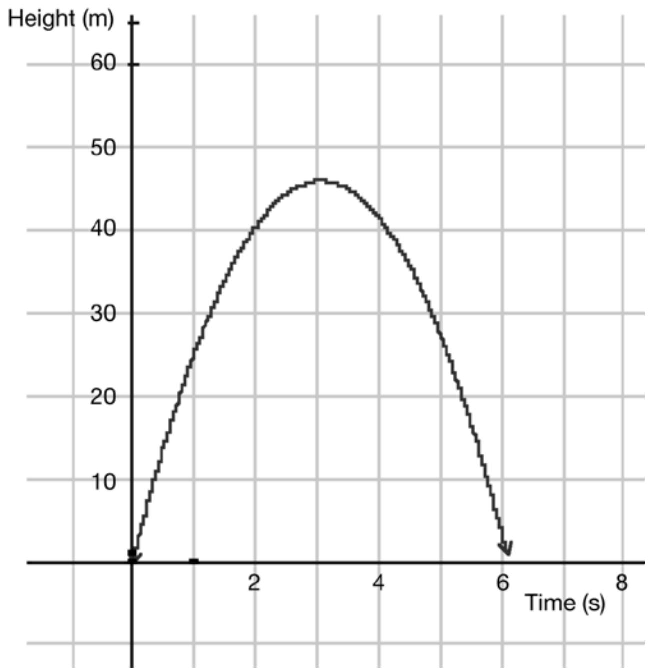
The point (7, –30.1) on the curve helps you sketch the curve even though you do not plot the point. A negative height makes no sense in the context of this question.



b) The average velocity of the ball between $t = 0$ and $t = 2$:

$$\frac{h(2) - h(0)}{2 - 0} = \frac{40.4 - 0}{2}$$
$$= 20.2 \text{ m/s}$$

c) To find when the ball hits the ground, solve for t when the height is 0. You can factor the equation:

$-4.9t^2 + 30t = -t(4.9t - 30) = 0$

The ball hits the ground when $4.9t - 30 = 0$,

$$t = \frac{30}{4.9} = 6.12 \text{ seconds}$$

d) Find the slope of the secant between $t = 1.4$ and $t = 1.5$:

$$\frac{h(1.5) - h(1.4)}{1.5 - 1.4} = \frac{33.975 - 32.395}{0.1}$$
$$= 15.79 \text{ m/s}$$

The instantaneous rate of change at $t = 1.5$ seconds is approximately 15.79 m/s.

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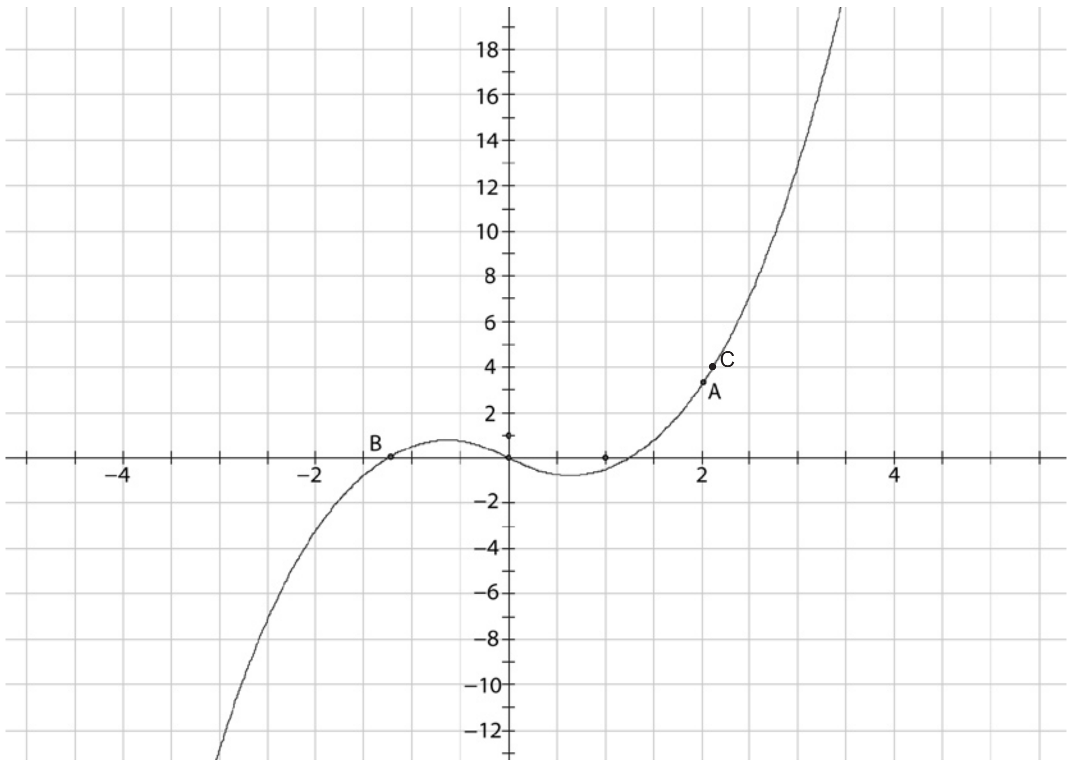
Note that your answer may be different if you choose a different interval.

3.



From the curve, you can approximate the coordinates of A to be (2, 3.3). Choose a point, $C(2.15, 4)$, on the curve close to A.

To find an approximation of the instantaneous rate at A, calculate the slope of the secant AC:

$$\frac{4 - 3.3}{2.15 - 2} = 4.67$$

The instantaneous rate of change at A is approximately 4.67.

Note that your answer may be different if you choose a different interval.

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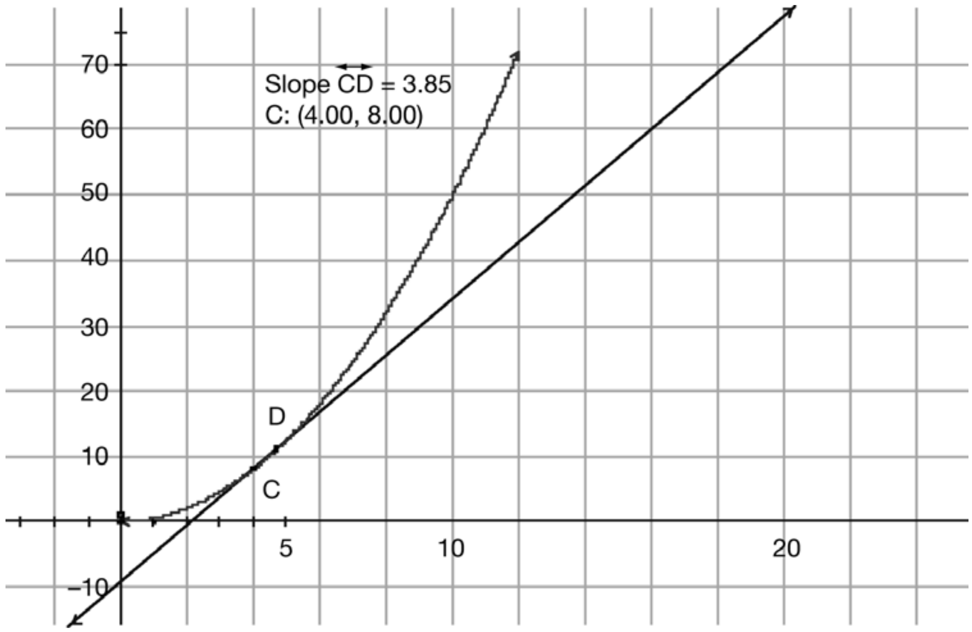
4. a)



b) Average rate of change: $\dfrac{d(6) - d(2)}{6 - 2} = \dfrac{0.5(6)^2 - 0.5(2)^2}{4}$

$= 4$ m/s

c) Instantaneous rate of change at $t = 3$ is 3 m/s.

d) Instantaneous rate of change at $t = 4$ is 4 m/s.

e) The cyclist is going faster at 4 seconds.

f) As d moves along the curve to the right, the slope CD increases. Therefore, the bike is going faster as the distance from the stop sign increases.

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Lesson 2

5.

n	sequence
1	$\frac{2}{1} = 2$
2	$\frac{3}{2} = 1.5$
3	$\frac{4}{3} = 1.33$
4	$\frac{5}{4} = 1.25$
5	$\frac{6}{5} = 1.2$
6	$\frac{7}{6} = 1.16$
7	$\frac{8}{7} = 1.14$
15	$\frac{16}{15} = 1.06$

The limiting value of the sequence is 1.

6.

n	sequence
1	1
2	0.666667
10	0.526316
11	0.52381
12	0.521739
17	0.515152
18	0.514286
19	0.513514
20	0.512821
21	0.512195
25	0.510204
30	0.508475
100	0.502513

The limiting value of the sequence is 0.5.

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7. The limiting value of the sequence is 4.

8. a) $\lim_{x\to 2} 3x^2 - 4 = 3(2)^2 - 4 = 8$

b) $\lim_{x\to 2} \frac{2}{x-3} = \frac{2}{2-3} = -2$

c) Substitute $x = -1$ into $\frac{x^2 - 3x - 4}{x^2 - 1}$. You get the indeterminate form $\frac{0}{0}$. Since the numerator and the denominator are both zero when $x = -1$, they should both have an $x + 1$ factor that you can divide out:

$$\lim_{x\to -1} \frac{x^2 - 3x - 4}{x^2 - 1} = \lim_{x\to -1} \frac{(x+1)(x-4)}{(x+1)(x-1)} = \lim_{x\to -1} \frac{(x-4)}{(x-1)} = \frac{-1-4}{-2} = \frac{5}{2}$$

d) If you substitute 0 into $\frac{\sqrt{4-x} - 2}{x}$, you get the indeterminate form $\frac{0}{0}$. Since this limit involves a square root, multiply and divide by the conjugate of $\sqrt{4-x} - 2$, which is $\sqrt{4-x} + 2$.

$$\lim_{x\to 0} \frac{\sqrt{4-x} - 2}{x} = \lim_{x\to 0} \frac{\left(\sqrt{4-x} - 2\right)\left(\sqrt{4-x} + 2\right)}{x\left(\sqrt{4-x} + 2\right)}$$

$$= \lim_{x\to 0} \frac{\left(\sqrt{4-x}\right)^2 - (2)^2}{x\left(\sqrt{4-x} + 2\right)} = \lim_{x\to 0} \frac{4 - x - 4}{x\left(\sqrt{4-x} + 2\right)}$$

$$= \lim_{x\to 0} \frac{-x}{x\left(\sqrt{4-x} + 2\right)} = \lim_{x\to 0} \frac{-1}{\left(\sqrt{4-x} + 2\right)} = \frac{-1}{4}$$

9. a) The slope of the secant is

$$\frac{f(3.5) - f(3)}{3.5 - 3} = \frac{3(3.5)^2 - 2(3.5) + 1 - (3(3)^2 - 2(3) + 1)}{0.5}$$

$$= \frac{30.75 - 22}{0.5}$$

$$= 17.5$$

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b) The average rate of change between $x = 3$ and $x = 3.5$ is the slope of the secant between $x = 3$ and $x = 3.5$, which is equal to 17.5.

c) $$\lim_{h\to 0}\frac{f(3+h)-f(3)}{h} = \lim_{h\to 0}\frac{3(3+h)^2-2(3+h)+1-22}{h}$$
$$= \lim_{h\to 0}\frac{3(9+6h+h^2)-6-2h+1-22}{h}$$
$$= \lim_{h\to 0}\frac{27+18h+3h^2-6-2h+1-22}{h}$$
$$= \lim_{h\to 0}\frac{3h^2+16h}{h}$$
$$= \lim_{h\to 0} 3h+16$$
$$= 16$$

d) The instantaneous rate of change of $f(x)$ at $x = 3$ is 16.

10. The instantaneous velocity is the instantaneous rate of change of $H(t)$ at $t = 1$ second:

$$\lim_{h\to 0}\frac{H(1+h)-H(1)}{h} = \lim_{h\to 0}\frac{-5(1+h)^2+50(1+h)+80-125}{h}$$
$$= \lim_{h\to 0}\frac{-5(1+2h+h^2)+50(1+h)+80-125}{h}$$
$$= \lim_{h\to 0}\frac{-5-10h-5h^2+50+50h+80-125}{h}$$
$$= \lim_{h\to 0}\frac{40h-5h^2}{h}$$
$$= \lim_{h\to 0} 40-5h$$
$$= 40$$

The instantaneous rate of change at $t = 1$ second is 40 m/s.

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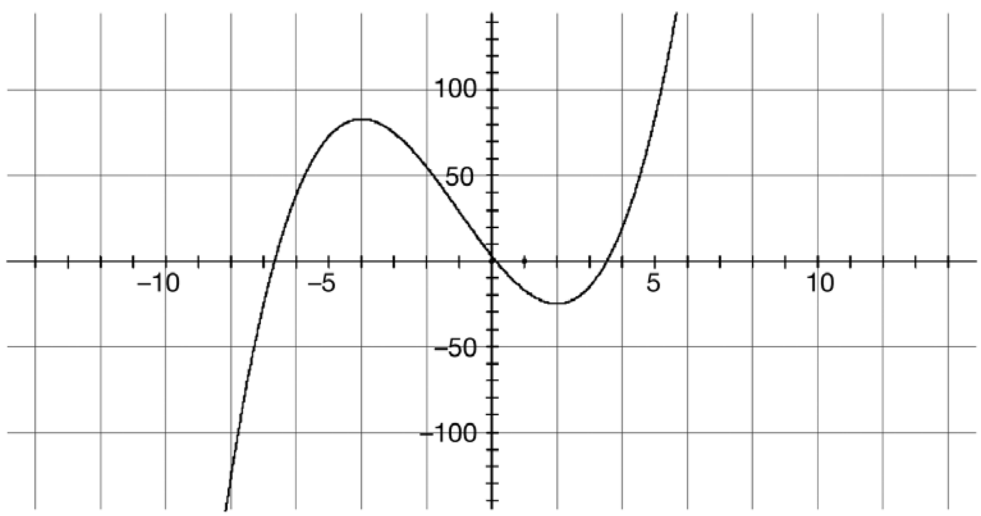
Lesson 3

11. a) The function is always increasing, therefore the instantaneous rate of change is positive for all x.

    b) The rate of change is zero at $x = 1$.

    c) In this case there are no local maximums or minimums, as the function is continuously increasing.

12. a) Use The Geometer's Sketchpad to get the graph of the function $f(x) = x^3 + 3x^2 - 24x + 3$:



The function is increasing when $x < -4$ and $x > 2$.

The function is decreasing when $-4 < x < 2$.

There is a local maximum at $x = -4$ and a local minimum at $x = 2$. This can be confirmed by finding the instantaneous rate of change at each point.

$f(-4 + h) - f(-4) = (-4 + h)^3 + 3(-4 + h)^2 - 24(-4 + h) + 3 - ((-4)^3 + 3(-4)^2 - 24(-4) + 3)$

$$= 48h - 12h^2 + h^3 - 24h + 3h^2 - 24h$$

$$= h^3 - 9h^2$$

$$\frac{f(-4+h) - f(-4)}{h} = \frac{h^3 - 9h^2}{h}$$

$$= h^2 - 9h$$

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As h goes to 0, the limit of $h^2 - 9h$ is 0.

Since you can see from the graph that the function is increasing when $x < -4$ and decreasing when $x > -4$, you can conclude that $x = -4$ is a local maximum.

At $x = 2$:

$f(2 + h) - f(2)$

$= (2 + h)^3 + 3(2 + h)^2 - 24(2 + h) + 3 - ((2)^3 + 3(2)^2 - 24(2) + 3)$

$= 8 + 12h + 6h^2 + h^3 + 3(4 + 4h + h^2) - 48 - 24h + 3 + 25$

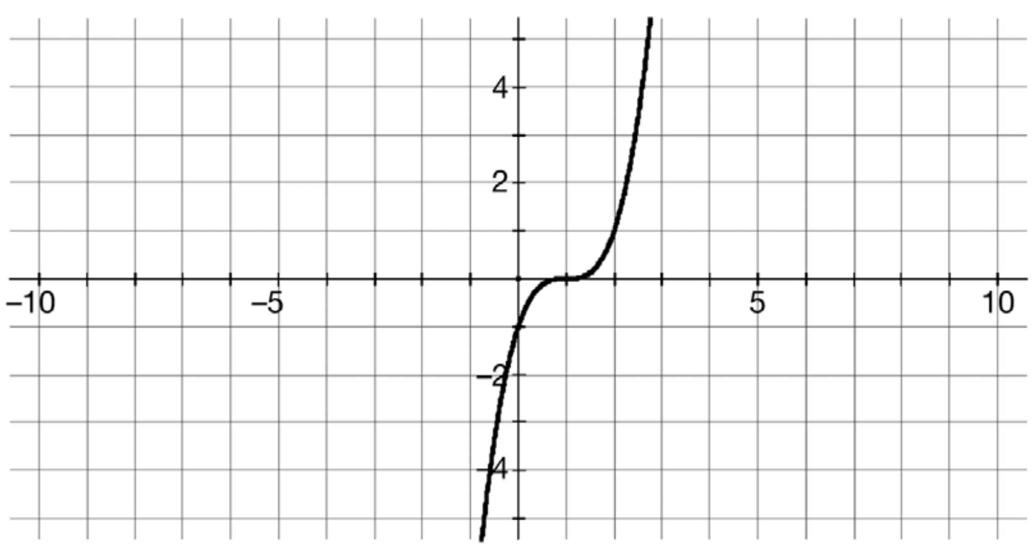
$= 9h^2 + h^3$

$$\frac{f(2+h) - f(2)}{h} = \frac{9h^2 + h^3}{h}$$
$$= 9h + h^2$$

As h goes to 0, the limit of $9h + h^2$ is 0.

Since you can see from the graph that the function is decreasing when $x < 2$ and increasing when $x > 2$, you can conclude that $x = 2$ is a local minimum.

b) Use The Geometer's Sketchpad to get the graph of the function $f(x) = x^3 - 3x^2 + 3x - 1$:



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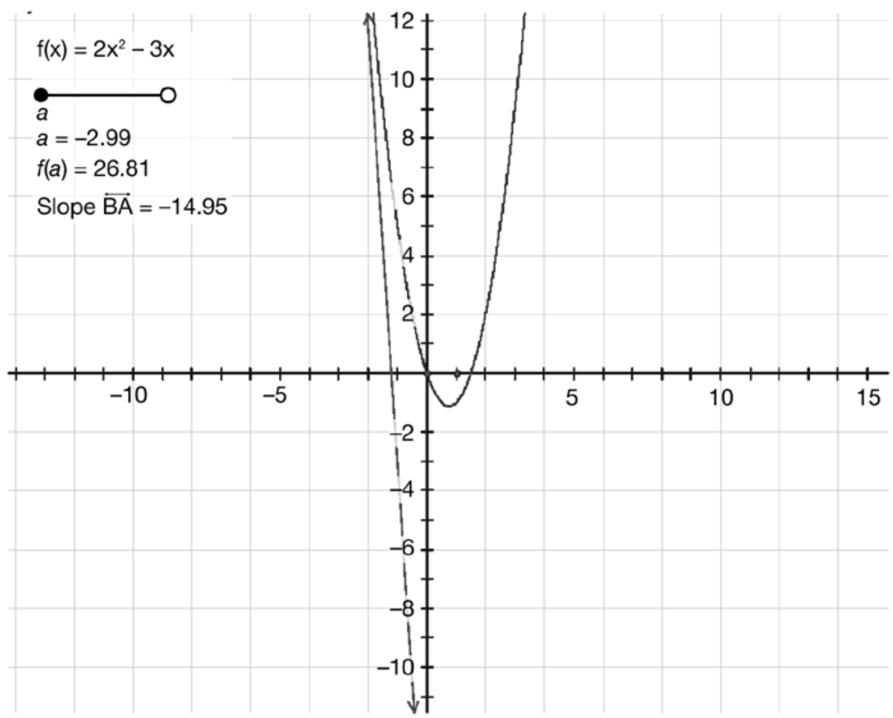
You can observe that the function is continuously increasing. The steepness of the increase varies, but it is always increasing.

The instantaneous rate of change at $x = 1$ is zero, but it is neither a local minimum nor a local maximum.

Lesson 4

13. a)



x	Slope of the tangent
–3	–15
–2	–11
–1	–7
0	–3
1	1
2	5
3	9

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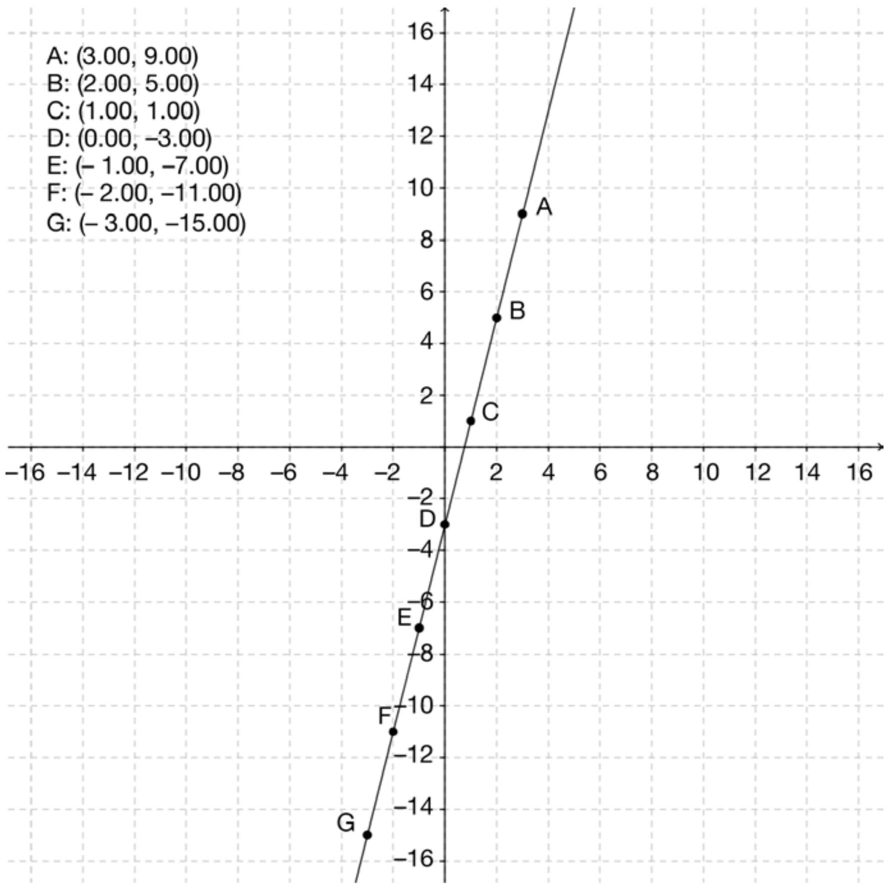
b)



Based on the graph, the derivative is a linear function.

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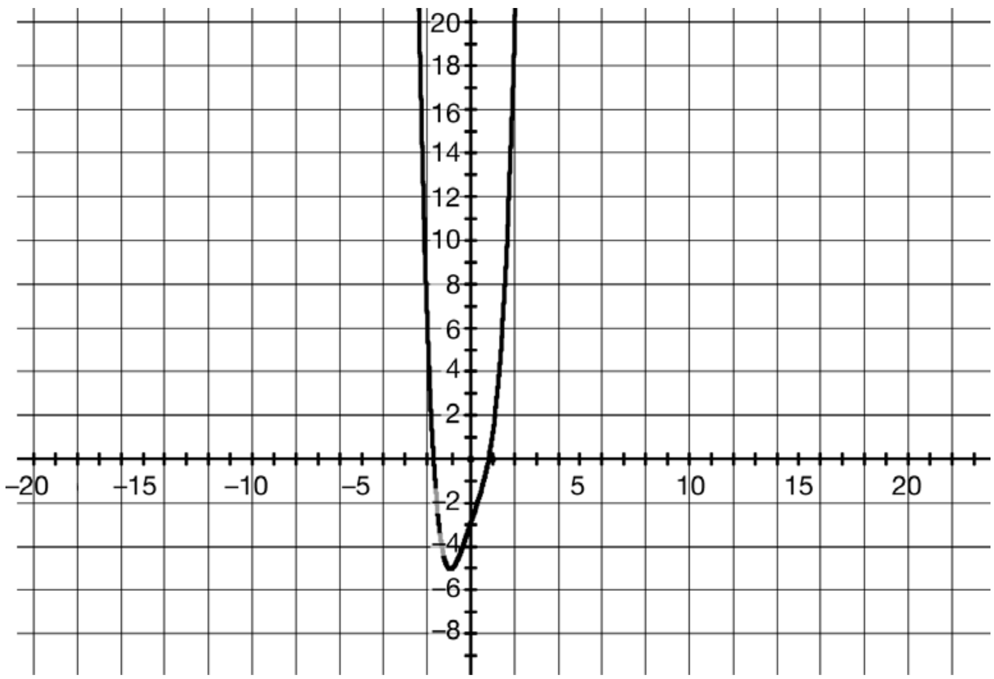
14. a)



x	$\frac{dy}{dx}$	First Difference	Second Difference	Third Difference
–2	–29			
–1	–1	–1 – (–29) = 28		
0	3	3 – (–1) = 4	4 – 28 = –24	
1	7	4	0	24
2	35	28	24	24

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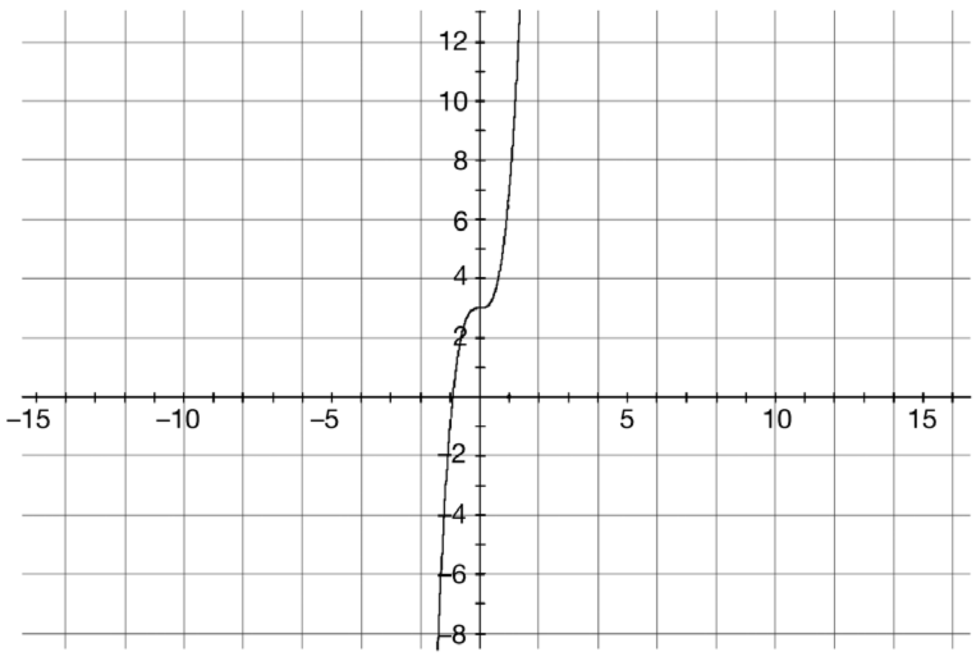
b)



The third differences are equal, therefore the derivative function is a polynomial of degree 3.

15. a)

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2(x+h)-2x}{h}\\ &= \frac{2x+2h-2x}{h}\\ &= 2\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}(2)\\ &= 2\end{aligned}$$

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b)
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-3(x+h)+2-x^2+3x-2}{h}\\ &= \frac{x^2+2xh+h^2-3x-3h+2-x^2+3x-2}{h}\\ &= \frac{2xh+h^2-3h}{h}\\ &= 2x+h-3\end{aligned}$$

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}(2x+h-3)\\ &= 2x-3\end{aligned}$$

16. a)
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{((\sqrt{x+h})^2-(\sqrt{x})^2)}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{(x+h-x)}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{h}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{1}{2\sqrt{x}}\end{aligned}$$

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b) $f(x) = \dfrac{1}{x-1}$

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\dfrac{1}{x+h-1} - \dfrac{1}{x-1}}{h} \\
&= \frac{\dfrac{x-1-(x+h-1)}{(x+h-1)(x-1)}}{h} \\
&= \frac{\dfrac{x-1-x-h+1}{(x+h-1)(x-1)}}{h} \\
&= \frac{\dfrac{-h}{(x+h-1)(x-1)}}{h} \\
&= \frac{-h}{h(x+h-1)(x-1)} \\
&= \frac{-1}{(x+h-1)(x-1)}
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)} &= \frac{-1}{(x-1)(x-1)} \\
&= \frac{-1}{(x-1)^2}
\end{aligned}$$

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Lesson 5

17. a) Use the power rule, $y' = 456x^{456-1} = 456x^{455}$

b) The derivative of a constant is 0 so $y' = 0$.

c) The constant multiple rule:

$$\begin{aligned} f'(x) &= -3(x^2)' \\ &= -3 \times 2x \\ &= -6x \end{aligned}$$

d) Rewrite the expression so you can use the power rule:

$$y = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$y' = \frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

e) Rewrite the expression so you can use the power rule:

$$\begin{aligned} y &= \sqrt{4x^3} \\ &= 2\sqrt{x^3} \\ &= 2(x^3)^{\frac{1}{2}} \\ &= 2x^{\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} y' &= 2\frac{3}{2}x^{\frac{3}{2}-1} \\ &= 3x^{\frac{1}{2}} \end{aligned}$$

18. a) $h'(x) = 15x^4 - 12x^2$

b) $$\begin{aligned} y' &= 3 \times 1.5x^{1.5-1} - 3 \\ &= 4.5x^{0.5} - 3 \end{aligned}$$

Suggested Answers to Support Questions

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19. The slope of the tangent to the curve at $x = a$ is the value of the derivative at a.

a) $y' = 3x^2 - 4x$

To find the slope, substitute $x = 2$ in the derivative.

The slope is $3(2)^2 - 4(2) = 4$.

b) $f(x) = (3x)^2 - 5x = 9x^2 - 5x$

$f'(x) = 18x - 5$

To find the slope, substitute $x = 0$ in the derivative.

The slope is $f'(0) = -5$

20. a) $y' = (x^2 - 3x)(4x^3 - 12x^2) + (2x - 3)(x^4 - 4x^3 - 5)$

b) $h'(x) = (x^4 - 4x^3 + x)(12x^5 - 20x^3 + 16x - 1) +$

$(4x^3 - 12x^2 + 1)(2x^6 - 5x^4 + 8x^2 - x)$

c)
$$\begin{aligned} y &= (\sqrt{x} - x^2 + x)(x^3 - 2x^2 + x) \\ &= (x^{\frac{1}{2}} - x^2 + x)(x^3 - 2x^2 + x) \end{aligned}$$

$$\begin{aligned} y' &= (x^{\frac{1}{2}} - x^2 + x)(3x^2 - 4x + 1) + (\frac{1}{2}x^{\frac{1}{2}-1} - 2x + 1)(x^3 - 2x^2 + x) \\ &= (x^{\frac{1}{2}} - x^2 + x)(3x^2 - 4x + 1) + (\frac{1}{2}x^{-\frac{1}{2}} - 2x + 1)(x^3 - 2x^2 + x) \end{aligned}$$

d)
$$\begin{aligned} h(t) &= \frac{t^3 - t + 1}{t^2} \\ &= (t^3 - t + 1)(t^{-2}) \end{aligned}$$

$$h'(t) = (t^3 - t + 1)(-2t^{-3}) + (3t^2 - 1)(t^{-2})$$

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21. a) $y' = 4(x^3 - 4x + 1)'(x^3 - 4x + 1)^3$

$= 4(3x^2 - 4)(x^3 - 4x + 1)^3$

b) $f(x) = x(x^3 + 2x)^{\frac{1}{2}}$

$f'(x) = x\left((x^3 + 2x)^{\frac{1}{2}}\right)' + (x^3 + 2x)^{\frac{1}{2}}$ * product rule

$= \frac{1}{2}x(3x^2 + 2)(x^3 + 2x)^{-\frac{1}{2}} + (x^3 + 2x)^{\frac{1}{2}}$

You can simplify this to obtain:

$f'(x) = \frac{x(3x^2 + 2)}{2\sqrt{x^3 + 2x}} + \sqrt{x^3 + 2x}$

22. a) The point on the curve is (0, 1).

$y' = 4x^3 - 3$, therefore the slope of the tangent is $4(0)^3 - 3 = -3$.

To find the y-intercept, substitute the coordinates of (0, 1).

$y = -3x + b$

$1 = -3(0) + b$

$1 = b$

The equation of the tangent is $y = -3x + 1$.

b) The point on the curve is (1, 4).

$y' = 12x^3 - 9x^2$, therefore the slope of the tangent is $12(1)^3 - 9(1)^2$, which is 3.

$y = 3x + b$

$4 = 3(1) + b$

$b = 1$

The equation of the tangent is $y = 3x + 1$.

Suggested Answers to Support Questions

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23. a) $y' = 3x^2 - 4x + 2$

Find the values of x where the derivative is 1.

$$3x^2 - 4x + 2 = 1$$
$$3x^2 - 4x + 1 = 0$$
$$(3x - 1)(x - 1) = 0$$

$x = \frac{1}{3}$ and $x = 1$

Case 1

The y-coordinate when $x = \frac{1}{3}$ is

$$\left(\frac{1}{3}\right)^3 - 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right) - 1 = \frac{1}{27} - \frac{2}{9} + \frac{2}{3} - 1$$
$$= \frac{1 - 6 + 18 - 27}{27}$$
$$= -\frac{14}{27}$$

$y = x + b$

Substitute the coordinates of $\left(\frac{1}{3}, -\frac{14}{27}\right)$ into $y = mx + b$ to find the y-intercept.

$$-\frac{14}{27} = \frac{1}{3} + b$$
$$b = -\frac{23}{27}$$

The equation is $y = x - \frac{23}{27}$.

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Case 2

The y-coordinate when $x = 1$ is 0.

$y = 1x + b$

$0 = 1 + b$

$b = -1$

The equation is $y = x - 1$.

There are two tangents to $y = x^3 - 2x^2 + 2x - 1$ with a slope equal to 1.

b) $y = -2x^2 + 4x$

$y' = -4x + 4$

Find the value of x when the slope equals 3:

$-4x + 4 = 3$

$-4x = -1$

$x = \dfrac{1}{4}$

The y-coordinate is

$$\begin{aligned} -2\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{4}\right) &= -\frac{1}{8} + 1 \\ &= \frac{7}{8} \end{aligned}$$

$y = 3x + b$

$\dfrac{7}{8} = \dfrac{3}{4} + b$

$b = \dfrac{1}{8}$

The equation of the tangent is $y = 3x + \dfrac{1}{8}$.

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