Department of Mathematics & Statistics

Course	Number	Section(s)
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2012	3
Instructors		**Course Examiners**
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Special Instructions

- **Only approved calculators are allowed.**
  For full marks you must show all your work.

MARKS

[10] **1.** (a) Suppose $f(x) = \sqrt[3]{x+1}$, and $g(x) = \dfrac{1}{x^3}$ Find $f \circ g$ and $g \circ f$.
Determine the domain and range of $f \circ g$ and $g \circ f$.

(b) Find the inverse of the function $f(x) = \sqrt[4]{3^x - 1}$. Find also the domain and range of f and f^{-1}.

c) Solve for x:

$$3^{\log_3(x^2)} = 5^{\log_5(2x)} + 2^3$$

[8] **2.** Evaluate the limits:

(a) $\displaystyle\lim_{x\to 3} \frac{\sqrt{4x-3}-3}{x^2-9}$ (b) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt[4]{(8x^6+2)(2x^3-4)^2}}{x(x^2-5)}$

Do not use l'Hôpital's rule.

[10] **3.** (a) Consider the function $f(x) = \dfrac{|x-1|}{x^2-5x+4}$.
Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find the value of a and b for which the function

$$f(x) = \begin{cases} 4-x & \text{if} \quad x < 0 \\ a\cos(\pi x) + b, & \text{if} \quad 0 \le x < 1 \\ -(x-1)^2 + 2, & \text{if} \quad x \ge 1 \end{cases}$$

is continuous at every point. Sketch the graph of this function.

[15] **4.** Find derivatives of the functions (you do not have to simplify the answer):

(a) $f(x) = \dfrac{(\sqrt{x} + \sqrt[4]{x})^2}{\ln(x)}$;

(b) $f(x) = x^{\pi} + \pi^{x}$;

(c) $f(x) = \cos^2(x)\arcsin(x)$;

(d) $f(x) = \sqrt{e^x + \sqrt{e^x + \sqrt{e^x}}}$;

(e) $f(x) = \sqrt{(x)^x}$ (use logarithmic differentiation).

[10] **5.** (a) The following limit represents the derivative of some function $f(x)$ at some number a.

$$\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$$

If $f(0) = 0$, what are $f(x)$ and a? Explain.

(b) Evaluate the derivative and find an equation of the tangent line to the graph $y = f(x)$ at $x = a$ of part (a) above.

(c) If $y = 4\sqrt{x}$, find the changes Δy and dy in y when x changes from 25 to 36.

[5] **6.** A function $P(t)$ representing the population of a colony of bacteria satisfies the differential equation $\dfrac{dP}{dt} = 2P$ (where t is time in hours), and the condition that $P = 5,000$ when $t = 0$. Find the function $P(t)$.

[10] **7.** (a) A curve is defined implicitly by the equation $\sin(x + y) = y^2 \cos x$. Verify that the point $(0,0)$ belongs to the curve. Find $\dfrac{dy}{dx}$ at this point.

(b) Explain (show) why the direct application of the l'Hôpital's rule to evaluate $\lim\limits_{t \to \pi/2} \dfrac{\sec(t)}{\tan(t)}$ does not work.
Then use another (simple) method that works and find the limit.

[6] **8.** (a) Let $f(x) = 1 + x/2 - \sqrt{1+x}$. Show that if $x > 0$ then $f'(x) > 0$

(b) Write the Mean Value Theorem formula for the function in part a), where the interval is $[0, x]$, and the point between 0 and x is c.

(c) Now use the formula in b) to show that $1 + x/2 > \sqrt{1+x}$ for all numbers $x > 0$.

[10] **9.** (a) The length of a rectangle is increasing at a rate of 5 cm/s and its width is increasing at a rate of 3 cm/s. How fast is the area of the rectangle increasing when the length is 20 cm and the width is 10 cm?

(b) A farmer wants to fence off a rectangular area of $10,500$ m^2 in a large field and then divide it in half with a fence parallel to one side of the rectangle. How can the farmer do this so as to minimize the cost of the fence?

[6] **10.** Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$. Find the absolute maximum and minimum values of $f(x)$ on $[-2, 3]$.

[10] **11.** Consider the following function and its derivatives:

$$f(x) = x\sqrt{2-x^2} \qquad f'(x) = \frac{2-2x^2}{\sqrt{2-x^2}} \qquad f''(x) = \frac{2x^3 - 6x}{(2-x^2)^{3/2}}$$

(a) Find the domain and check for symmetry. Find all horizontal and vertical asymptotes (if any).

(b) Find the interval(s) where the function is increasing, interval(s) where the function is decreasing, and local maxima and minima (if any).

(c) Find the interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Suppose a function f is the same as its inverse function f^{-1}. For example if $f(x) = -x + 2$ then $f^{-1}(x) = f(x)$. In this example f is a **decreasing** function. Prove that if $f = f^{-1}$ and f is an **increasing** function, then $f(x) = x$ for all x. [*Hint:* show that both $f(x) < x$ and $f(x) > x$ are not possible].