MCV4U-A

20

Intersections

Introduction

In previous lessons, you learned that a linear equation of three variables represents a plane. You also learned that solving a system of linear equations with three variables is equivalent to finding the intersection of the planes they represent.

In this lesson, you will begin by finding the solutions to systems of two and three linear equations and interpreting your solution geometrically. You will then apply the different techniques and properties of vectors to solve problems involving distance and intersections of lines and planes.

Once you have completed this lesson and studied for the Final Test, you may take a Practice Test. It will help you prepare for the Final Test.

The Practice Test is on your course page on the ILC website. When you have completed it, you can use the Practice Test Suggested Answers on your course page to check your work. Follow the instructions at the end of this lesson.

Estimated Hours for Completing This Lesson	
Intersections	2.5
Putting It Together	1.5
Key Questions	1

What You Will Learn

After completing this lesson, you will be able to

- determine the intersection of two lines in three-space
- determine the solution of a linear system of equations with three variables and interpret the solution geometrically
- use the projection of a vector onto a direction vector to determine distance

Intersections

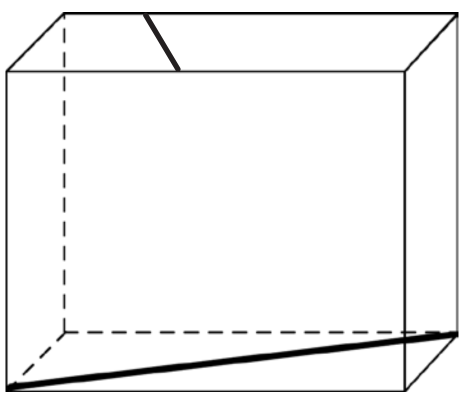
In this section, you will solve algebraically for the intersections of two lines, of a line and a plane, and of two planes.

Intersection of Two Lines in Three-Space

Two lines in three-space can be described as

- coincident
- parallel
- intersecting in one point
- skew

Skew lines are two lines that do not intersect but are not parallel. An example would be a diagonal on the bottom of a rectangular box and the indicated line on the top.



Examples

Find the intersection of each of the following pairs of lines:

a) $\vec{l_1} = (1, -2, 4) + t(5, -1, -3)$ and $\vec{l_2} = (3, 6, 7) + s(3, -2, -4)$

b) $\vec{l_1} = (-1, 2, 3) + t(25, 5, -10)$ and $\vec{l_2} = (1, 3, 5) + s(5, 1, -2)$

c) $\vec{l_1} = (3, 1, 0) + t(5, 1, 3)$ and $\vec{l_2} = (4, 10, -9) + s(-1, 2, -3)$

Solutions

a) $\vec{l_1} = (1, -2, 4) + t(5, -1, -3)$ and $\vec{l_2} = (3, 6, 7) + s(3, -2, -4)$

First, check if the lines have the same direction:

$\vec{d_1} = (5, -1, -3)$ is a direction vector of $\vec{l_1}$ and $\vec{d_2} = (3, -2, -4)$ is a direction vector of $\vec{l_2}$.

Observe that the ratios of the components of the direction vectors are not equal:

$$\frac{5}{3} \neq \frac{-1}{-2} \neq \frac{-3}{-4}$$

This means that the two do not have the same direction and they either intersect at a point or they are skew.

Next, write the equation of the lines in parametric form:

$\vec{l_1}$	$\vec{l_2}$
$x = 1 + 5t$	$x = 3 + 3s$
$y = -2 - t$	$y = 6 - 2s$
$z = 4 - 3t$	$z = 7 - 4s$

A point on both lines must satisfy the parametric equations of the two lines; so, if the lines intersect, you should be able to find values of s and t that satisfy the following linear system.

$1 + 5t = 3 + 3s$ **[1]**

$-2 - t = 6 - 2s$ **[2]**

$4 - 3t = 7 - 4s$ **[3]**

(**Note**: The numbers in square brackets are labels for each equation.)

Use the elimination method to solve the linear system. You saw the elimination method in previous grades when you solved linear systems. The idea is to eliminate one of the variables by multiplying the equations with a constant and adding or subtracting.

Start with any two of the three equations, since you have only two variables.

Simplify equations [1] and [2] to get the following:

$5t - 3s = 2$ **[4]**

$-t + 2s = 8$ **[5]**

Now you want to eliminate one of the variables, such as the s variable. Multiply equation [4] times 2 and equation [5] times 3 to get the following:

$[4] \times 2 = 10t - 6s = 4$ **[6]**

$[5] \times 3 = -3t + 6s = 24$ **[7]**

Add the two equations to solve for t.

$$10t - 3t - 6s + 6s = 4 + 24$$
$$7t = 28$$
$$t = 4$$

Substitute $t = 4$ into [5].

$$-4 + 2s = 8$$
$$2s = 12$$
$$s = 6$$

Finally, substitute $t = 4$ and $s = 6$ into both sides of [3] to check for consistency:

Left Side (LS):

$$4 - 3t = 4 - 3(4)$$
$$= -8$$

Right Side (RS):

$$7 - 4s = 7 - 4(6)$$
$$= -17$$

The two values are not the same. You can conclude that the two lines are skew.

b) $\vec{l_1} = (-1, 2, 3) + t(25, 5, -10)$ and $\vec{l_2} = (1, 3, 5) + s(5, 1, -2)$

$\vec{d_1} = (25, 5, -10)$ and $\vec{d_2} = (5, 1, -2)$ are direction vectors of the two lines.

Observe that the components of $\vec{d_1}$ and $\vec{d_2}$ have the same ratio:

$$\frac{25}{5} = \frac{5}{1} = \frac{-10}{-2} = 5$$

Hence, the two lines are either parallel or coincident. To check if the two lines are coincident, it is sufficient to show that they have one point in common. The point $(-1, 2, 3)$ is on line 1; is it also on line 2? Can you find a value of s that satisfies the following equations?

$-1 = 1 + 5s$ **[1]**

$2 = 3 + s$ **[2]**

$3 = 5 - 2s$ **[3]**

Solve for s:

[1]
$$\begin{aligned} -1 &= 1 + 5s \\ -2 &= 5s \\ s &= -\frac{2}{5} \end{aligned}$$

[2]
$$\begin{aligned} 2 &= 3 + s \\ s &= -1 \end{aligned}$$

Since the two values are not equal, you can conclude that the two lines are parallel and distinct.

c) $\vec{l_1} = (3, 1, 0) + t(5, 1, 3)$ and $\vec{l_2} = (4, 10, -9) + s(-1, 2, -3)$

Observe that the two lines do not have the same direction because the ratios of the components of the direction vectors are not equal:

$$\frac{5}{-1} \neq \frac{1}{2} \neq \frac{3}{-3}$$

Therefore, the lines either intersect or they are skew.

The parametric equations of the two lines:

$\vec{l_1}$	$\vec{l_2}$
$x = 3 + 5t$	$x = 4 - s$
$y = 1 + t$	$y = 10 + 2s$
$z = 3t$	$z = -9 - 3s$

A point on the two lines must satisfy both sets of parametric equations:

Equate the coordinates:

$3 + 5t = 4 - s$ **[1]**

$1 + t = 10 + 2s$ **[2]**

$3t = -9 - 3s$ **[3]**

Simplify the equations:

$5t + s = 1$ **[4]**

$t - 2s = 9$ **[5]**

$3t + 3s = -9$ **[6]**

Now you can use [4] and [5] to solve for s and t:

[4] × 2 $\quad 10t + 2s = 2$

$\quad t - 2s = 9$

Add the equations together:

$$11t = 11$$
$$t = 1$$

Substitute $t = 1$ into [4] to solve for s:

$$s = 1 - 5t$$
$$s = -4$$

Now use $s = -4$ and $t = 1$ to check if equation [3] is satisfied:

LS:	RS:
$3t = 3(1)$	$-9 - 3s = -9 - 3(-4)$
$= 3$	$= 3$

The third equation is satisfied and hence the two lines intersect at a unique point. You can now find the coordinates of the intersection point by substituting $t = 1$ into $\vec{l_1}$ or $s = -4$ into $\vec{l_2}$.

Using $\vec{l_1}$:

$x = 3 + 5t = 8$

$y = 1 + t = 2$

$z = 3t = 3$

The point of intersection of the two lines is (8, 2, 3). If you check $\vec{l_2}$ you will find that you get the same values.

Support Question
(do not send in for evaluation)

23. Find the intersection of each pair of lines:

a) $\vec{l_1} = (1, 3, 2) + t(1, -2, 4)$ and $\vec{l_2} = (2, 1, 8) + s(1, -2, 6)$

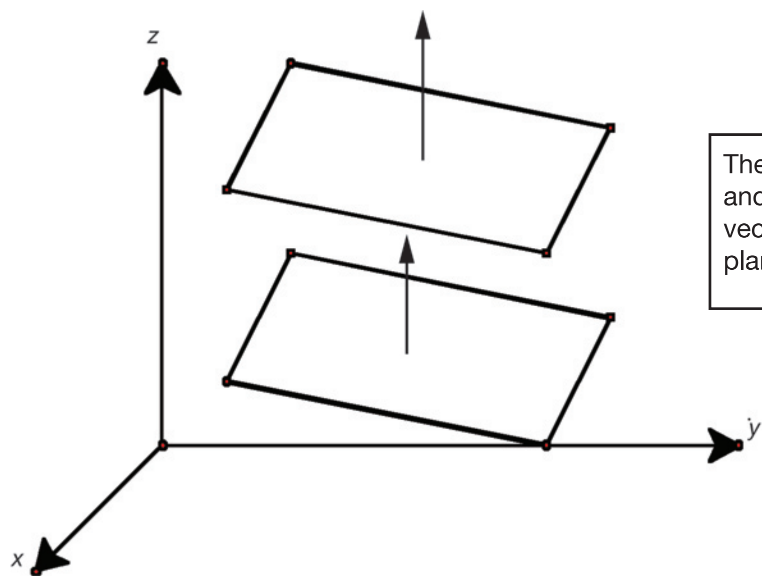
b) $\vec{l_1} = (2, 1, 0) + t(2, 4, -10)$ and $\vec{l_2} = (11, 2, -1) + s(-8, -16, 40)$

There are Suggested Answers to Support Questions at the end of this unit.

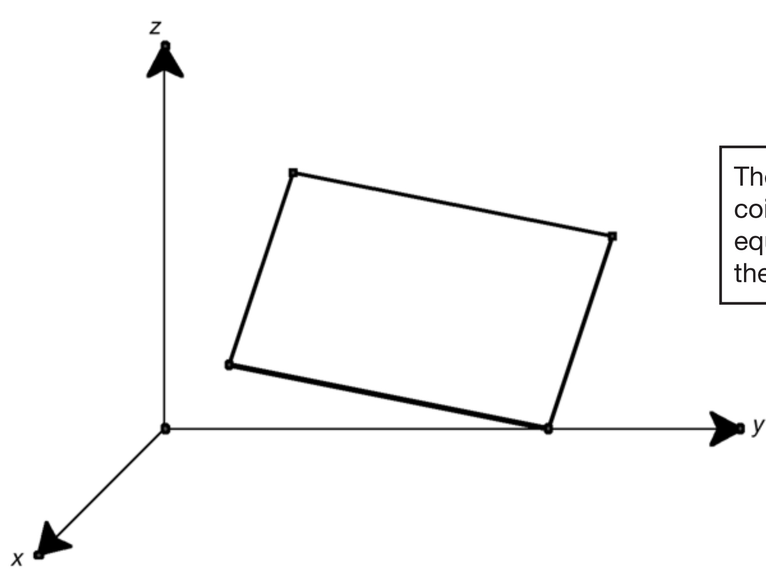
Intersection of Two Planes

In Lesson 18, you explored the ways in which two planes intersect. In this section, you will solve for the intersection algebraically.

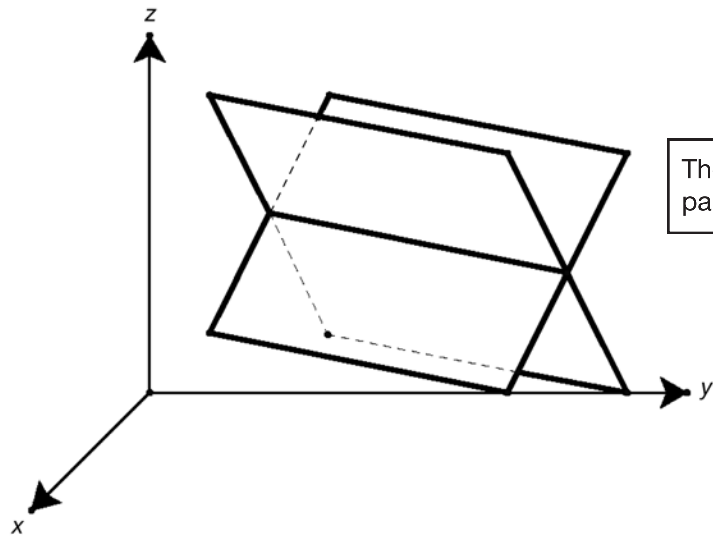
Recall that there are three cases for the intersection of two planes: the planes can be parallel, they can be coincident, or they can intersect in a line.



The two planes are parallel and distinct. The normal vectors are collinear. The planes do not intersect.



The two planes are coincident. Both equations given represent the same plane.



The two planes are not parallel and intersect at a line.

Examples

Determine the intersection of each of the two planes:

a) $2x - y + 4z = 12$ and $6x - 3y + 12z = -9$

b) $2x - y + 3z = 1$ and $x - 3y + 12z = -4$

c) $x + y + 4z = 1$ and $3x + 3y + 12z = 3$

Solutions

a) Label the planes P_1: $2x - y + 4z = 12$ and P_2: $6x - 3y + 12z = -9$.

The normal to P_1 is $\vec{n}_1 = (2, -1, 4)$ and the normal to P_2 is $\vec{n}_2 = (6, -3, 12)$. The two normals are scalar multiples of each other, $\vec{n}_2 = 3\vec{n}_1$, so the two normal vectors are collinear. Therefore, the two planes are either parallel or coincident.

The constant values in the equations of the planes do not have the same ratio:

$-9 \neq 3(12)$

Therefore, you can conclude that the two planes are not coincident but parallel, and hence they have no points of intersection.

You can also conclude that the linear system has no solution.

b) $2x - y + 3z = 1$ and $x - 3y + 12z = -4$

The normals to the planes are $\vec{n}_1 = (2, -1, 3)$ and $\vec{n}_2 = (1, -3, 12)$.

The two vectors are not multiples of each other, so the planes are not parallel and must intersect in a line. (**Note**: If you do not see that the normal vectors are not multiples of each other, take the ratio of the corresponding components. You will notice that they are not the same: $\frac{2}{1} \neq \frac{-1}{-3} \neq \frac{3}{12}$).

To determine the line of intersection, use the elimination method:

$2x - y + 3z = 1$ **[1]**

$x - 3y + 12z = -4$ **[2]**

Since you have $2x$ in equation [1], multiply equation [2] by 2 to obtain [3]:

$2x - y + 3z = 1$ **[1]**

[2] × 2 $2x - 6y + 24z = -8$ **[3]**

Subtract [1] and [3]

$$2x - y + 3z - (2x - 6y + 24z) = 1 - (-8)$$

$$2x - 2x - y + 6y + 3z - 24z = 9$$

$$5y - 21z = 9 \quad \textbf{[4]}$$

Write y in terms of z:

$5y - 21z = 9$

$5y = 9 + 21z$

$y = \frac{9}{5} + \frac{21}{5}z$

Use either [1] or [2] to find x in terms of z. Use [2] for this example.

$$x - 3y + 12z = -4$$

$$x - 3\left(\frac{9}{5} + \frac{21}{5}z\right) + 12z = -4$$

$$x - \frac{27}{5} - \frac{63}{5}z + 12z = -4$$

$$x = \frac{63}{5}z - 12z + \frac{27}{5} - 4$$

$$x = \frac{3}{5}z + \frac{7}{5}$$

Now let $z = t$ and write x and y in terms of t to get the parametric equations of the line of intersection of the two planes:

$$x = \frac{3}{5}t + \frac{7}{5}$$

$$y = \frac{9}{5} + \frac{21}{5}t$$

$$z = t$$

c) $x + y + 4z = 1$ and $3x + 3y + 12z = 3$

The normal to P_1 is $\vec{n}_1 = (1, 1, 4)$ and the normal to P_2 is $\vec{n}_2 =$ $(3, 3, 12)$. The two normals are scalar multiples of each other, $\vec{n}_2 = 3\vec{n}_1$, so the two normals are collinear. Notice that the constant values have the same ratio, $3 = 3(1)$. Therefore, you can conclude that the two planes are coincident.

Support Questions

(do not send in for evaluation)

24. Find the parametric equations of the intersection line of the two planes $2x - 3y - z + 1 = 0$ and $3x - 2y + 3z - 4 = 0$

25. Find the equation of the line that passes through the point $A(3, -1, 2)$ and is parallel to the intersection line of the two planes $x + y + 3z - 10 = 0$ and $6x - 2y + z - 10 = 0$.
(**Hint**: Recall that you need a direction vector to find the equation of a line.)

Intersection of Three Planes

In Lesson 18, you explored the various ways in which three planes can intersect. Now you will solve for those types of intersections algebraically.

Given three planes with normal vectors $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$, in order to identify the nature of the intersection of the three planes you need to calculate $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)$.

If:

- $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \neq 0$, the normal vectors are not coplanar, so there is a single point of intersection
- $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$, the normal vectors are coplanar and there may or may not be points of intersection

Example

Solve the linear system:

$x - y + 2z - 2 = 0$

$2x + y - z + 1 = 0$

$x + y + z - 1 = 0$

Solution

Use the elimination method to solve for the intersection. Start by labelling the equations [1], [2], and [3]:

$x - y + 2z - 2 = 0$ **[1]**

$2x + y - z + 1 = 0$ **[2]**

$x + y + z - 1 = 0$ **[3]**

Choose a variable to eliminate. In this case, eliminate the y variable by adding [1] and [2,] and, [1] and [3]:

$x - y + 2z - 2 = 0$	**[1]**
$2x + y - z + 1 = 0$	**[2]**
$3x + z - 1 = 0$	**[4]**

$x - y + 2z - 2 = 0$	**[1]**
$x + y + z - 1 = 0$	**[3]**
$2x + 3z - 3 = 0$	**[5]**

By doing this, you reduce the system of three equations with three unknowns to a system of two equations with two unknowns:

$3x + z - 1 = 0$ **[4]**

$2x + 3z - 3 = 0$ **[5]**

Multiply [4] by 3 and subtract the two equations to solve for x.

$$\begin{array}{ll} [4] \times 3 & 9x + 3z - 3 = 0 \\ & \underline{2x + 3z - 3 = 0} \\ & 7x = 0 \\ & x = 0 \end{array}$$

Substitute $x = 0$ into [4] to solve for z.

$$\begin{aligned} 3x + z - 1 &= 0 \\ 3(0) + z - 1 &= 0 \\ z &= 1 \end{aligned}$$

Substitute $x = 0$ and $z = 1$ into [1] to find y.

$$\begin{aligned} x - y + 2z - 2 &= 0 \\ 0 - y + 2 - 2 &= 0 \\ y &= 0 \end{aligned}$$

The three planes intersect at the point (0, 0, 1).

The normal vectors to the planes:

$\vec{n}_1 = (1, -1, 2)$, $\vec{n}_2 = (2, 1, -1)$, and $\vec{n}_3 = (1, 1, 1)$

You can confirm that the three planes intersect at a point by finding the value of $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)$:

$$\begin{array}{cccc} 1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array}$$

$(1)(1) - (1)(-1) = 2$

$(-1)(1) - (1)(2) = -3$

$(2)(1) - (1)(1) = 1$

$$\vec{n}_2 \times \vec{n}_3 = (2, -3, 1)$$

$$\begin{aligned}\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) &= (1, -1, 2) \cdot (2, -3, 1)\\ &= (1)(2) + (-1)(-3) + (2)(1)\\ &= 7\\ &\neq 0\end{aligned}$$

The planes are not coplanar, so the planes intersect in a single point.

Example

Solve the following linear system and interpret the solution geometrically:

$x + 3y + 3z = 8$ **[1]**

$x - y + 3z = 4$ **[2]**

$2x + 6y + 6z = 16$ **[3]**

Solution

The normal vectors are $\vec{n}_1 = (1, 3, 3)$, $\vec{n}_2 = (1, -1, 3)$, and $\vec{n}_3 = (2, 6, 6)$.

Observe that equation [3] is 2 × [1]. Therefore, the two planes are coincident. Observe also that $\vec{n}_2$ is not parallel to $\vec{n}_1$ and $\vec{n}_3$. You can conclude that two of the three planes are coincident. The intersection of three planes is a line. To solve for the line of intersection, solve the following linear system:

[1] $x + 3y + 3z = 8$

[2] $x - y + 3z = 4$

Subtract the equations to solve for y:

$$4y = 4$$

$$y = 1$$

Substitute $y = 1$ into [1]:

$$x + 3 + 3z = 8$$

$$x + 3z = 5$$

$$x = 5 - 3z$$

To find the parametric equations, set $z = t$ and solve for x:

$x = 5 - 3t$

Geometrically, you can conclude that the planes intersect along the line defined by the following parametric equations:

$x = 5 - 3t$

$y = 1$

$z = t$

Example

Solve the system of the linear equations and interpret your solution geometrically:

$x - y + z + 2 = 0$

$2x - 2y - z = 0$

$4x - 4y + z + 3 = 0$

Solution

Use the elimination method to solve the system:

$x - y + z + 2 = 0$ **[1]**

$2x - 2y - z = 0$ **[2]**

$4x - 4y + z + 3 = 0$ **[3]**

Add [1] and [2] to eliminate z:

$$\begin{array}{lr} x - y + z + 2 = 0 & \textbf{[1]} \\ 2x - 2y - z = 0 & \textbf{[2]} \\ \hline 3x - 3y + 2 = 0 & \textbf{[4]} \end{array}$$

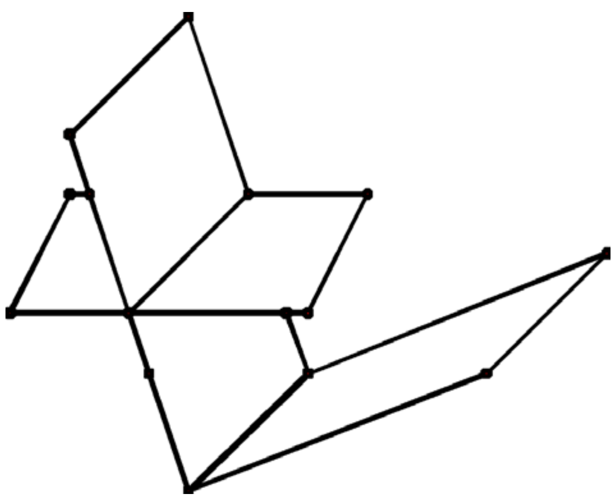
Subtract [3] from [1] to eliminate z:

$$\begin{array}{lr} x - y + z + 2 = 0 & \textbf{[1]} \\ \underline{4x - 4y + z + 3 = 0} & \underline{\textbf{[3]}} \\ -3x + 3y - 1 = 0 & \textbf{[5]} \end{array}$$

You now have two equations with two unknowns. Add equations [4] and [5] to eliminate another variable:

$$\begin{array}{lr} 3x - 3y + 2 = 0 & \textbf{[4]} \\ \underline{-3x + 3y - 1 = 0} & \underline{\textbf{[5]}} \\ 1 = 0 & \end{array}$$

Since all unknowns have been eliminated, you can conclude that there are no points that satisfy the three equations. Geometrically, you have the case that is illustrated in the following diagram.



Support Questions
(do not send in for evaluation)

26. Solve the following system of linear equations and interpret your solution geometrically:

$3x + z + 11 = 0$

$2x + y + z + 4 = 0$

$x + y + z - 3 = 0$

27. Solve the following system of linear equations and interpret your solution geometrically:

$x - 2y - 2z - 6 = 0$

$2x - 5y + 3z + 10 = 0$

$3x - 7y + z + 9 = 0$

Putting It Together

In this section, you will use properties and techniques that you learned in the vectors portion of the course to calculate distance in a number of different scenarios.

The Distance Between a Point and a Line

Example

Find the equation of the line drawn from point $A(0, 1, 3)$ to the plane $\vec{v} = (0, 0, 2) + t(-1, 1, 3) + s(2, 0, -3)$ and determine the distance from point A to the plane.

Solution

The plane has two direction vectors: $\vec{a} = (-1, 1, 3)$ and $\vec{b} = (2, 0, -3)$.

A line perpendicular to the plane has the same direction as the normal to the plane. The normal to the plane can be found by finding the cross product of $\vec{a}$ and $\vec{b}$:

$\vec{a} \times \vec{b} = (-1, 1, 3) \times (2, 0, -3) = (-3, 3, -2)$

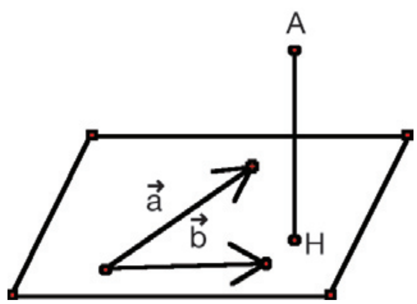
The parametric equations of a line with direction $(-3, 3, -2)$ that passes through A:

$x = -3t$

$y = 1 + 3t$

$z = 3 - 2t$

Here is a diagram that includes the point A and the relevant plane:



You can approach the second part of the question in two different ways.

Approach 1

Find the distance between the intersection point of the line through A and the plane. To solve for the intersection, substitute the components of the parametric equations of the line into the scalar equation of the plane.

The scalar equation of the plane is $-3x + 3y - 2z + d = 0$. You can solve for d by substituting the coordinates of any point on the plane given by $\vec{v} = (0, 0, 2) + t(-1, 1, 3) + s(2, 0, -3)$. Use $(0, 0, 2)$:

$-3(0) + 3(0) - 2(2) + d = 0$

$d = 4$

The scalar equation of the plane is $-3x + 3y - 2z + 4 = 0$.

A point on the line has coordinate $(-3t, 1 + 3t, 3 - 2t)$ for some t. If the point is on the plane, it must also satisfy the scalar equation of the plane:

$-3(-3t) + 3(1 + 3t) - 2(3 - 2t) + 4 = 0$

$9t + 3 + 9t - 6 + 4t + 4 = 0$

$22t = -1$

$t = -\dfrac{1}{22}$

The intersection point of the line and the plane:

$$\begin{aligned}(-3t, 1 + 3t, 3 - 2t) &= \left(\frac{3}{22}, 1 - \frac{3}{22}, 3 + \frac{2}{22}\right)\\ &= \left(\frac{3}{22}, \frac{19}{22}, \frac{68}{22}\right)\end{aligned}$$

The distance between the two points:

$$\vec{AH} = \vec{OH} - \vec{OA}$$

$$= \left(\frac{3}{22}, \frac{19}{22}, \frac{68}{22}\right) - (0, 1, 3)$$

$$= \left(\frac{3}{22}, -\frac{3}{22}, \frac{2}{22}\right)$$

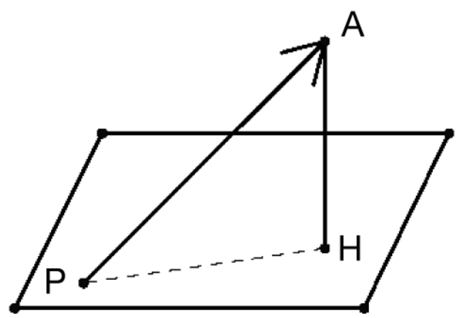
$$|\vec{AH}| = \sqrt{\left(\frac{3}{22}\right)^2 + \left(-\frac{3}{22}\right)^2 + \left(\frac{2}{22}\right)^2}$$

$$\begin{aligned}\left|\vec{AH}\right| &= \sqrt{\frac{22}{22^2}}\\ &= \frac{\sqrt{22}}{22}\end{aligned}$$

Approach 2

As you saw in Approach 1, the scalar equation of the plane is $-3x + 3y - 2z + 4 = 0$. Let P be the point (0, 0, 2).

As shown in the following diagram, the projection of $\vec{PA}$ onto the normal to the plane is $\vec{AH}$ and the magnitude of $\vec{AH}$ is the distance from point A to the plane.



Recall from Lesson 17 that $proj_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$.

$$\begin{aligned}
\left|\vec{AH}\right| &= \left|proj_{\vec{n}}(\vec{PA})\right| \\
&= \left|\frac{\vec{PA}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}\right| \\
&= \left|\frac{\vec{PA}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\right| |\vec{n}|
\end{aligned}$$

$$\begin{aligned}
\vec{PA} &= \vec{OA} - \vec{OP} \\
&= (0,1,3) - (0,0,2) \\
&= (0,1,1)
\end{aligned}$$

$$\begin{aligned}
\left|\vec{AH}\right| &= \left|\frac{(0,1,1)\cdot(-3,3,\ -2)}{(-3,3,\ -2)\cdot(-3,3,\ -2)}\right|\sqrt{(-3)^2 + (3)^2 + (-2)^2} \\
&= \left|\frac{3-2}{(9+9+4)}\right|\sqrt{9+9+4} \\
&= \frac{\sqrt{22}}{22}
\end{aligned}$$

Both approaches give the same answer. You should be comfortable using both approaches to solve this type of problem.

Support Question
(do not send in for evaluation)

28. Find the distance from the point $A(-2, 1, 2)$ to the plane $3x - 2y + 5z - 2 = 0$.

Distance Between Two Skew Lines

The following example illustrates how to find the distance between two skew lines.

Example

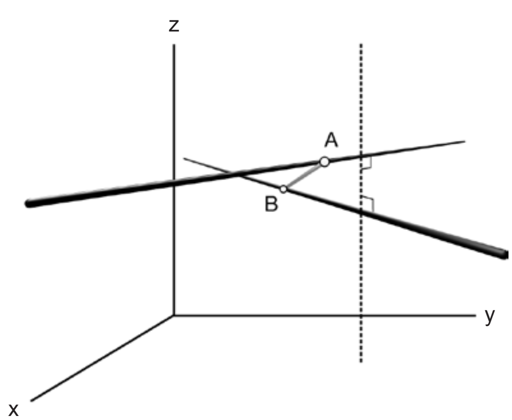
Find the distance between the two skew lines $\vec{l_1}$ and $\vec{l_2}$:

$\vec{l_1} = (1, -2, 4) + t(5, -1, -3)$ and $\vec{l_2} = (3, 6, 7) + s(3, -2, -4)$

Solution

The shortest distance between the skew lines is parallel to their common perpendicular. The magnitude of any vector connecting two points on the two lines projected on to a perpendicular to the lines is the shortest distance between the skew lines.

This is illustrated in the following diagram. The vertical dashed line is a common perpendicular to the two skew lines.



The projection of $\vec{AB}$ onto the normal is the distance between the two skew lines.

$A(1, -2, 4)$ is a point on $\vec{l_1}$ and $B(3, 6, 7)$ is a point on $\vec{l_2}$.
$\vec{AB} = \vec{OB} - \vec{OA} = (2, 8, 3)$

$\vec{d_1} = (5, -1, -3)$ is a direction vector of $\vec{l_1}$ and $\vec{d_2} = (3, -2, -4)$ is a direction vector of $\vec{l_2}$.

$\vec{d_1} \times \vec{d_2}$ is perpendicular to both lines. Using the usual procedure you find:
$\vec{n_1} = \vec{d_1} \times \vec{d_2} = (-2, 11, -7)$

The distance between $\vec{l_1}$ and $\vec{l_2}$ is $\left|proj_{\vec{n}}\left(\vec{AB}\right)\right|$.

$$\left|proj_{\vec{n}}\left(\vec{AB}\right)\right|$$

$$= \left|\frac{\vec{AB}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n}\right|$$

$$= \left|\frac{\vec{AB}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\right| |\,\vec{n}\,|$$

$$= \left|\frac{(2, 8, 3)\cdot(-2, 11, -7)}{(-2, 11, -7)\cdot(-2, 11, -7)}\right|\sqrt{(-2)^2 + (11)^2 + (-7)^2}$$

$$= \left|\frac{(2)(-2) + (8)(11) + (3)(-7)}{(4 + 121 + 49)}\right|\sqrt{4 + 121 + 49}$$

$$= \frac{63}{174}\sqrt{174}$$

Support Questions

(do not send in for evaluation)

29. Calculate the distance from point $A(1, -2, 3)$ to the plane $2x - 3y + 6z - 8 = 0$.
30. Find the distance between the two skew lines $\vec{l_1} = (-6, -4, 0) + t(0, 1, -1)$ and $\vec{l_2} = (0, 5, 3) + s(2, 0, -1)$.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(18 marks)

59. Find the equation of the line that passes through the point $A(1, -4, 2)$ and is parallel to the intersection line of the two planes $x - 2y + 3z - 1 = 0$ and $x - 4y + 2z - 8 = 0$. **(6 marks)**
60. Solve the system of the linear equations and interpret your solution geometrically: **(8 marks)**

    $2x + y + 2z - 4 = 0$

    $x - y - z - 2 = 0$

    $x + 2y - 6z - 12 = 0$
61. Find the distance from the point $A(-2, 1, 2)$ to the plane $3x - 2y + 5z + 10 = 0$. **(4 marks)**

Congratulations! You have completed all of the lessons in the course. Now do the Reflection for Unit 4. Follow any other instructions you have received from ILC about submitting your coursework, then send it to ILC. A teacher will mark your work and you will receive your results online.

Prepare for the Final Test as you wait for your results. ILC will get in touch with you about writing the test. To prepare for it, review the course material and do the Practice Test.

Practice Test

To help prepare you for the Final Test, there is a Practice Test on your course page on the ILC website.

The Practice Test is organized in the same way as the Final Test, so you will know what to expect. The front page is like the one you will see when you write the Final Test. Read over the instructions so that you are familiar with them, and so that there are no surprises at the time of the real test.

You will have two hours to write the Final Test. Time yourself when you do this Practice Test to see if you can complete the work in two hours. At the beginning of each section, you are told how long that section should take you to complete.

To access the Practice Test, go to your course page on the ILC website.

When you have completed the Practice Test, check the suggested answers on your course page to see how well you have done.

If you had trouble with any of the questions on the Practice Test, go back to that section of the course and review the material and Support Questions carefully.

If you have trouble completing the test in two hours, it may be because you are not familiar with some of the ideas you need to know. Plan to review the course material fully before the Final Test.

Do not send the Practice Test in to ILC for marking. It is for your own use, as practice for the real thing!