MCV4U-A

19

Lines and Planes Using Scalar, Vector, and Parametric Equations

Introduction

There are various ways to express algebraically a line in two-dimensional space or a plane in three-dimensional space, specifically with scalar, vector, or parametric equations. These types of equations have many advanced applications. For example, parametric equations can be used in weather forecasting for the marine environment.

In this lesson, you will be introduced to the three types of equations in two-space and three-space, as well as to how to convert between the various ways of representing a line and a plane.

Estimated Hours for Completing This Lesson	
Lines in Two-Space	1
Lines in Three-Space	1
Planes in Three-Space	2
Key Questions	1

What You Will Learn

After completing this lesson, you will be able to

- identify a scalar equation of a line in two-space
- represent a line in two-space using a vector equation or parametric equations
- convert between a scalar equation, a vector equation, and parametric equations of a line in two-space
- identify a scalar equation of a plane in three-space
- explain the relationship between a scalar equation of a plane and a normal vector to the plane
- represent a plane in three-space using a vector equation or parametric equations
- convert between a scalar equation, a vector equation, and parametric equations of a plane in three-space

Lines in Two-Space

You have previously learned that an equation of the form $ax + by + c = 0$, where a, b, and c are constants, is called a linear equation, and its graph in two-space is a straight line. For example, the graph of $2x + 5y - 10 = 0$ is a straight line in two-space.

In this lesson, you will learn about other ways that a line in two-space can be described algebraically.

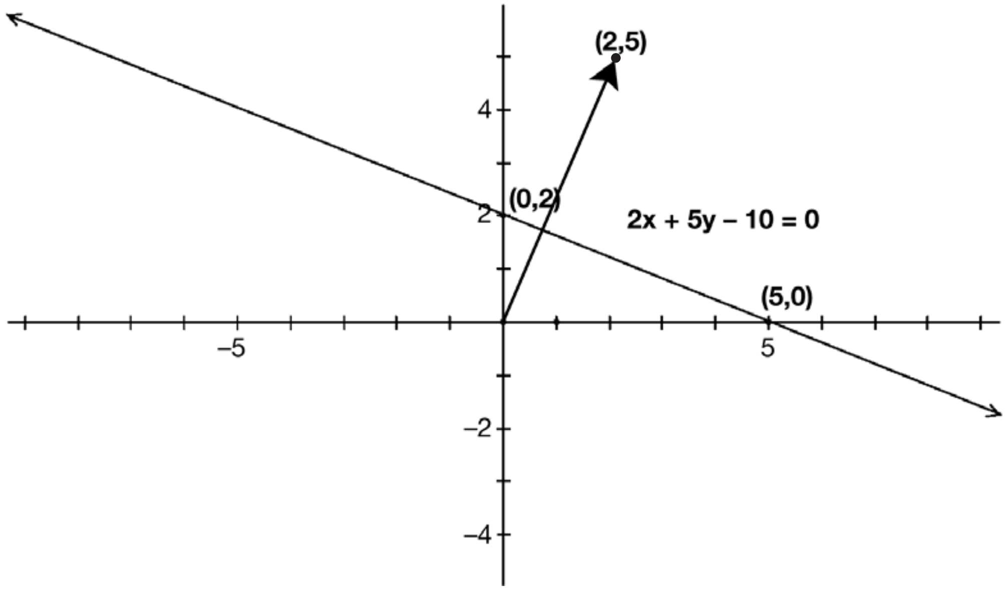
Scalar Equation of a Line in Two-Space

An equation of the form $ax + by + c = 0$ is also called a scalar equation of a straight line. This name is used because the quantities a, b, c, x, and y represent scalars and not vectors. For example, the equation $2x + 5y - 10 = 0$ is a scalar equation for a particular line in two-space. The equation $4x + 10y - 20 = 0$ is another scalar equation for the same line, and $3x - 7y + 18 = 0$ is a scalar equation for a different line.

It turns out that, given a scalar equation for a line, it's easy to identify a vector perpendicular to the line. Later in this lesson, you'll see that something similar happens with planes.

A line in two-space with a scalar equation $ax + by + c = 0$ is perpendicular to the vector (a, b). For example, as illustrated in the following diagram, the line with scalar equation $2x + 5y - 10 = 0$ is perpendicular to the vector $(2, 5)$.

Notice that if you wish to draw the graph of $2x + 5y - 10 = 0$, one possible approach is to rewrite the equation in slope-intercept form. Another approach is to notice that two points are enough to determine a straight line. You can find particular points on the line by substituting values for x and finding y or vice versa. The number 0 is a nice round number to use: you find that $(5, 0)$ and $(0, 2)$ are two examples of points on the line:



Vector Equation of a Line in Two-Space

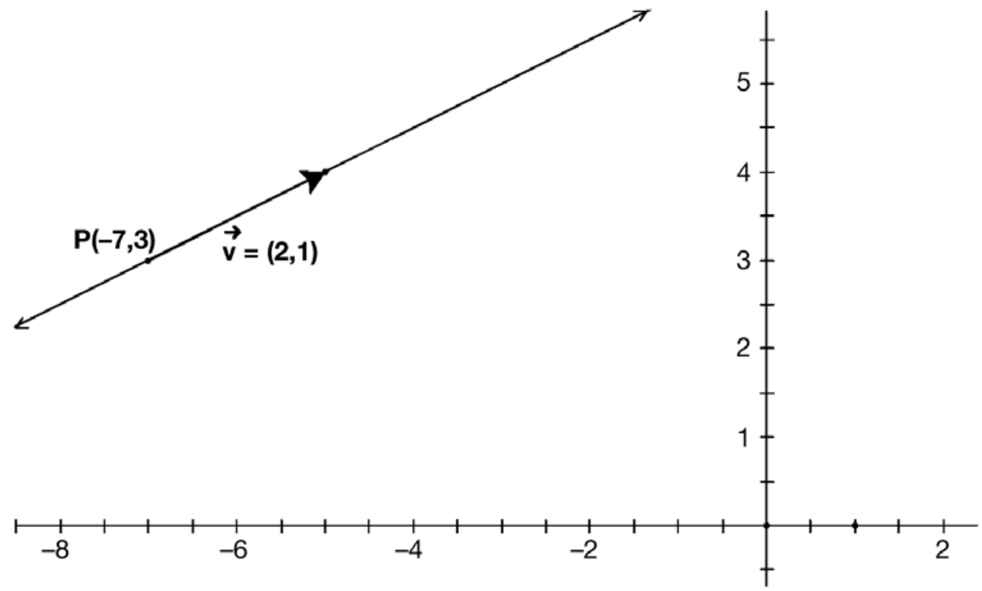
There are several different ways to describe a straight line geometrically. As you saw earlier, two points are enough to determine a straight line. Another way to describe a straight line is to provide one point on the line and a vector that describes the direction of the line. This vector is called a direction vector for the line.

Example

Draw a graph of the line through the point $P(-7, 3)$ with direction vector $\vec{v} = (2, 1)$.

Solution

The line is graphed by starting from point P and drawing the direction vector.



An example of a vector equation for this line is $(x,y) = (-7, 3) + t(2, 1)$, where t represents a real number that can have any value: positive, negative, or zero. When you start at the point $P(-7, 3)$ and add any scalar multiple of the vector $(2, 1)$, you will get a point on the line. Conversely, starting at any point on the line you can reach $P(-7, 3)$ by adding an appropriate multiple of the direction vector $(2, 1)$.

When you write a vector equation for a line, many different direction vectors are possible. Any scalar multiple of the direction vector can be used to represent the same line. For example, equally correct direction vectors for the line you just looked at would be $(6, 3)$, $(20, 10)$, $(-2, -1)$, $(1, 0.5)$, and equally correct vector equations for that line would include the following:

$(x,y) = (-7, 3) + t(6, 3)$

$(x,y) = (-7, 3) + t(20, 10)$

$(x,y) = (-7, 3) + t(-2, -1)$

$(x,y) = (-7, 3) + t(1, 0.5)$

Furthermore, the point (–7, 3) can be replaced with any particular point on the line, such as (–5, 4) or (–3, 5). The other correct vector equations for the given line would include these:

$(x,y) = (-7, 3) + t(2, 1)$

$(x,y) = (-5, 4) + t(2, 1)$

$(x,y) = (-3, 5) + t(2, 1)$

In general, the following is true:

> A **vector equation** of a straight line in two-space has the form $(x, y) = (x_0, y_0) + t(a, b)$ where (x_0, y_0) is a particular point on the line and (a, b) is a direction vector for the line.

Parametric Equations of a Line in Two-Space

Closely related to a vector equation for a line are parametric equations of a line, where you have an equation for each coordinate.

Example

In a previous example, you considered the line with vector equation $(x,y) = (-7, 3) + t(2, 1)$. This equation can also be written like this:

$(x,y) = (-7, 3) + t(2, 1) = (-7, 3) + (2t, t) = (-7 + 2t, 3 + t)$

You can separate this into the following pair of equations:

$x = -7 + 2t$

$y = 3 + t$

This latter pair of equations is an example of parametric equations for a line in two-space. In general, the following applies:

> **Parametric equations** of a straight line in two-space have this form:
>
> $x = x_0 + at$
>
> $y = y_0 + bt$
>
> (x_0, y_0) is a particular point on the line, (a, b) is a direction vector for the line, and t can be any real number.

Since there are several different forms in which a straight line in two-space can be expressed algebraically, you can switch back and forth between the various forms.

Examples

Write a vector equation and parametric equations for the following:

a) The line through $A(-3, 5)$ and $B(2, 7)$

b) The line defined by the equation $4x - 5y = 8$

Solutions

a) One thing you need is a direction vector for the line. An example of a direction vector is $\vec{AB} = \vec{OA} - \vec{OB} = (2, 7) - (-3, 5) = (5, 2)$. You also need one point on the line. You have been given two, so you may choose one arbitrarily. Suppose you choose A.

A vector equation for the line is $(x, y) = (-3, 5) + t(5, 2)$

The parametric equations for the line are

$x = -3 + 5t$

$y = 5 + 2t$

b) To answer this, you again need one point on the line and a direction vector. As before, you can find a direction vector if you know two points. Find points on the line by substituting a value for x or y. For example, you can substitute either $x = 0$ or $y = 0$, since those are easy values to use.

Substituting $x = 0$ gives $-5y = 8$, so $(0, -\frac{8}{5})$ is a point on the line. Substituting $y = 0$ gives $4x = 8$, so $(2, 0)$ is a point on the line. An example of a direction vector for the line is $(2, 0) - (0, -\frac{8}{5}) = (2, \frac{8}{5})$. (Note that $(0, -\frac{8}{5}) - (2, 0)$ would be equally correct since this vector is also parallel to the line.)

A vector equation for the line is $(x, y) = (2, 0) + t(2, \frac{8}{5})$

The parametric equations for the line:

$x = 2 + 2t$

$y = 0 + (\frac{8}{5})t.$

Support Question
(do not send in for evaluation)

19. Write a vector equation and a parametric equation for the following:

a) The line through $A(-1, 2)$ and $B(-3, 4)$

b) The line defined by the equation $2x + 3y = 10$

There are Suggested Answers to Support Questions at the end of this unit.

Lines in Three-Space

A single linear equation in two-space represents a straight line (called a "scalar equation" for the line). You've already seen that, in three-space, a single linear equation represents not a line but a plane. It is still possible to describe a straight line in three-space algebraically, but in three-space there is no such thing as a scalar equation of a line.

A Line as the Intersection of Two Planes

You observed in Lesson 18 that in three-dimensional space, two planes that are not parallel and not coincident will intersect in a line.

If you're given a pair of linear equations in three-space, each equation represents a plane. If the two planes are not parallel and not coincident, you know the points that satisfy both equations must form a straight line in three-space. How do you tell if the two planes are parallel? How do you describe a line in three-space? You will develop answers to these questions in the remainder of this lesson.

Vector Equation of a Line in Three-Space

In three-space, just as in two-space, you can describe a line algebraically if you know the coordinates of one point on the line and a direction vector for the line. The main difference is that, in three-space, a point has three coordinates and a vector has three components.

For example, in three-space:

- the line through $A(2, -3, 6)$ parallel to $\vec{v} = (3, 1, -7)$ has vector equation $(x, y, z) = (2, -3, 6) + t(3, 1, -7)$
- the line through $A(2, -3, 6)$ parallel to the y-axis has vector equation $(x, y, z) = (2, -3, 6) + t(0, 1, 0)$, since the vector $(0, 1, 0)$ points in the direction of the y-axis

In general, you have the following:

A **vector equation** of a straight line in three-space has the form $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$ where (x_0, y_0, z_0) is a particular point on the line, (a, b, c) is a direction vector for the line, and t can be any real number.

Parametric Equation of a Line in Three-Space

Just as you did in two-space, you can take a vector equation for a line in three-space and separate it into one equation for each coordinate.

Earlier, you saw the line with vector equation $(x, y, z) = (2, -3, 6) + t(3, 1, -7)$.

You can rewrite this equation:

$(x, y, z) = (2, -3, 6) + t(3, 1, -7) = (2, -3, 6) + (3t, t, -7t) = (2 + 3t, -3 + t, 6 - 7t)$

This can be separated into the three equations:

$x = 2 + 3t$

$y = -3 + t$

$z = 6 - 7t$

In general, the following applies:

Parametric equations of a straight line in three-space have this form:

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

(x_0, y_0, z_0) is a particular point on the line, (a, b, c) is a direction vector for the line, and t can be any real number.

As with two-space, you can switch back and forth between the various ways to express a straight line in three-space algebraically.

Example

Write a vector equation and a parametric equation for the line that goes through the points $A(4, 1, -7)$ and $B(-2, 5, -8)$.

Solution

You need a direction vector for the line. Use either $\vec{AB}$ or $\vec{BA}$ (or any scalar multiple of either of those vectors). To solve this example, use $\vec{BA} = \vec{OA} - \vec{OB} = (4, 1, -7) - (-2, 5, -8) = (6, -4, 1)$. You also need one point on the line, but you have two to choose from. Use A.

A vector equation for the line:

$(x, y, z) = (4, 1, -7) + t(6, -4, 1)$

Parametric equations for the line:

$x = 4 + 6t$

$y = 1 - 4t$

$z = -7 + t$

Support Question
(do not send in for evaluation)

20. Write a vector equation and a parametric equation for the following:

a) The line through $A(1, -2, 2)$ and parallel to $\vec{u} = (1, -1, 1)$

b) The line through $A(-1, 0, 3)$ and parallel to the x-axis

c) The line through $A(-1, 2, 3)$ and $B(1, 3, 4)$

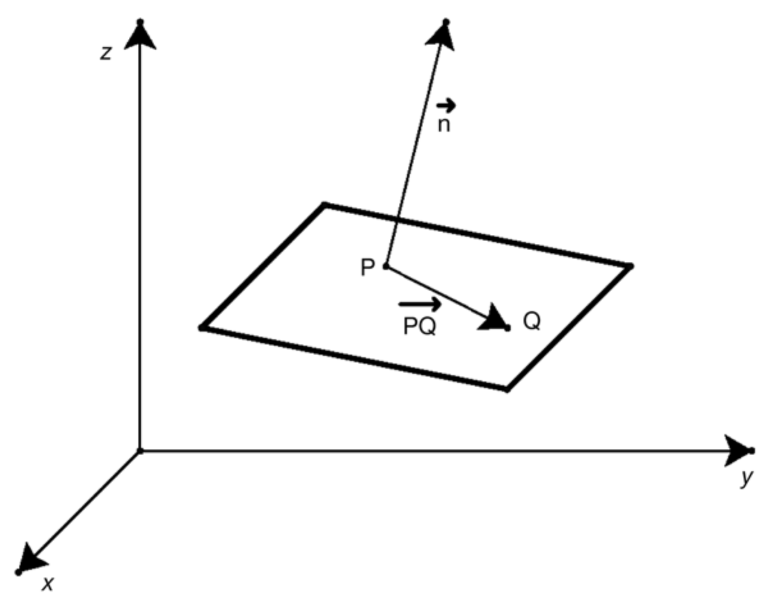
Planes in Three-Space

You know that in three-space a linear equation means any equation of the form $ax + by + cz + d = 0$, where a, b, c, and d are constants, and the graph of such an equation in three-space is a plane. For example, the graph of the equation $x + 2y + 4z - 12 = 0$ is a plane in three-space.

Normal Vector to a Plane

Given a plane in three-space, to say that a vector or a line is normal to the plane simply means that it is perpendicular to the plane. Imagine a plane in three-space and imagine a line or vector perpendicular to that plane. You will notice that being perpendicular to the plane is equivalent to being perpendicular to every line or vector in the plane.

The symbol $\vec{n}$ is often used to denote a normal vector, although other variable names may also be used. In the following diagram, P and Q are points in the plane, which means that $\overrightarrow{PQ}$ is a vector in the plane, and hence $\vec{n}$ is perpendicular to $\overrightarrow{PQ}$.



Here is an important fact about normal vectors and planes:

> The plane in three-space whose equation is given by $ax + by + cz + d = 0$ has normal vector (a, b, c).

Notice that normal vectors are not unique. Any scalar multiple of a normal vector to a plane (where the scalar may be positive or negative) is another normal vector to the plane. For example, the plane in three-space with equation $x + 2y + 4z - 12 = 0$ has $\vec{n} = (1, 2, 4)$ as a normal vector.

Why is this true? Notice that if $\vec{n}$ is normal to the plane, then $\vec{n}$ must be normal to every vector in the plane. If $P = (u, v, w)$ and $Q = (x, y, z)$ are two points in the plane, then $\vec{PQ} = (x - u, y - v, z - w)$ is a vector in the plane.

Since (u, v, w) and (x, y, z) are both points in the plane, they must satisfy the equation of the plane:

$u + 2v + 4w - 12 = 0$

$x + 2y + 4z - 12 = 0$

Subtracting these two equations gives

$(x - u) + 2(y - v) + 4(z - w) = 0$

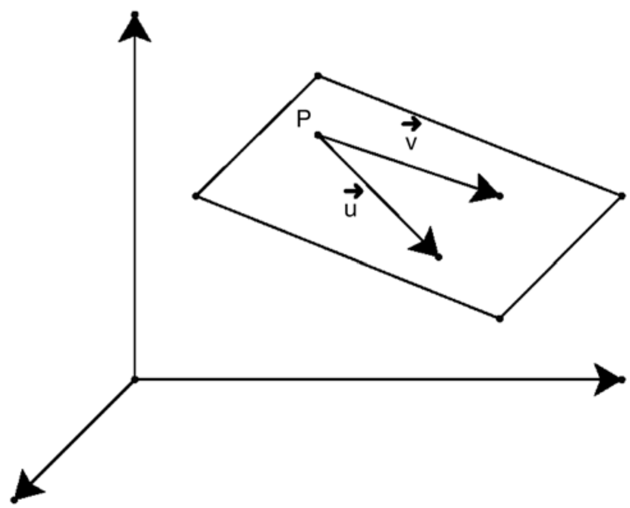
If you think about dot product, this statement might look familiar. It is precisely the statement that the vectors $(1, 2, 4)$ and $(x - u, y - v, z - w)$ have dot product zero and hence are perpendicular. Therefore, $\vec{n}$ is normal to every vector in the plane.

Scalar Equation of a Plane in Three-Space

As you saw earlier, in three-space, any equation of the form $ax + by + cz + d = 0$, where a, b, c, and d are constants, represents a plane. This equation is called a scalar equation of the plane. There are also other possible ways of algebraically representing a plane in three-space.

Vector Equation of a Plane in Three-Space

Recall that for a vector equation of a line in three-space, you need one point on the line and a direction vector for the line. A plane can be described similarly using a vector equation, but the difference is that a vector equation for a plane requires one point on the plane and a pair of non-collinear direction vectors. The situation is illustrated by the following picture:



A **vector equation** of a plane in three-space has the following form:

$(x, y, z) = (x_0, y_0, z_0) + s(a_1, a_2, a_3) + t(b_1, b_2, b_3)$

(x_0, y_0, z_0) is a particular point on the plane, (a_1, a_2, a_3) and (b_1, b_2, b_3) are two non-collinear direction vectors for the plane, and s and t can be any real numbers.

Parametric Equation of a Plane in Three-Space

Much as you did with equations of a line in three-space, you can take a vector equation of a plane in three-space and separate it into one equation for each coordinate. For example, the plane through the point (2, –7, 3) that contains the vectors (–1, 8, 5) and (3, 9, –2) has this vector equation:

$(x, y, z) = (2, -7, 3) + s(-1, 8, 5) + t(3, 9, -2)$

This can also be written as

$(x, y, z) = (2, -7, 3) + (-s, 8s, 5s) + (3t, 9t, -2t) = (2 - s + 3t,$
$-7 + 8s + 9t, 3 + 5s - 2t)$

This can be separated into the three equations:

$x = 2 - s + 3t$

$y = -7 + 8s + 9t$

$z = 3 + 5s - 2t$

In general, you have the following:

> **Parametric equations** of a plane in three-space have this form:
>
> $x = x_0 + sa_1 + tb_1$
>
> $y = y_0 + sa_2 + tb_2$
>
> $z = z_0 + sa_3 + tb_3$
>
> (x_0, y_0, z_0) is a particular point on the plane, (a_1, a_2, a_3) and (b_1, b_2, b_3) are two non-collinear direction vectors for the plane, and s and t can be any real numbers.

In the rest of this lesson, you will focus on converting between different ways that a plane in three-space can be expressed algebraically. You can describe a plane by specifying three points on the plane, or by specifying a point and two direction vectors, or by specifying a point and a normal vector.

Example

Determine scalar, vector, and parametric equations for the plane that passes through the points $A = (1, 2, 3)$, $B = (1, 4, 5)$, and $C = (2, 4, 6)$.

Solution

Due to the simple relationship between the scalar equation and the normal vector, a good way to obtain a scalar equation for a plane is first to find a normal vector. You can obtain a normal vector by taking the cross product of any two vectors in the plane (you learned in an earlier lesson that the cross product of $\vec{u}$ and $\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$).

Since A, B, and C are points in the plane, $\vec{AB}$ and $\vec{AC}$ are examples of vectors in the plane.

$\vec{AB} = (1, 4, 5) - (1, 2, 3) = (0, 2, 2)$

$\vec{AC} = (2, 4, 6) - (1, 2, 3) = (1, 2, 3)$

You can now compute $\vec{AB} \times \vec{AC}$ as follows:

$$\begin{array}{cccc} 2 & 2 & 0 & 2 \\ 2 & 3 & 1 & 2 \end{array}$$

$(2)(3) - (2)(2) = 2$

$(2)(1) - (3)(0) = 2$

$(0)(2) - (1)(2) = -2$

$\vec{AB} \times \vec{AC} = (2, 2, -2)$. The normal vector can then be used to form a scalar equation for the plane:

$2x + 2y - 2z + d = 0$

You need to find the value of d. You can do this by substituting any one of the given points A, B, or C. For example, substituting A gives you the following:

$2(1) + 2(2) - 2(3) + d = 0$

$2 + 4 - 6 + d = 0$

$d = 0$ and the scalar equation of the plane is $2x + 2y - 2z = 0$

To double-check, you can verify that all three of the given points A, B, and C satisfy this equation. You'll find that they do.

Since you have at least one point on the plane as well as two direction vectors, $\vec{AB}$ and $\vec{AC}$, you can also write a vector equation of the plane (note that $\vec{AB}$ and $\vec{AC}$ are not collinear because neither is a scalar multiple of the other):

$(x, y, z) = (1, 2, 3) + s(0, 2, 2) + t(1, 2, 3)$

You can also separate this equation into three equations, thus giving parametric equations of the plane:

$x = 1 + t$

$y = 2 + 2s + 2t$

$z = 3 + 2s + 3t$

Support Question

(do not send in for evaluation)

21. Determine scalar, vector, and parametric equations for the plane that passes through the points $A = (3, 2, 5)$, $B = (0, -2, 2)$, and $C = (1, 3, 1)$.

The next example will help you find the scalar form and the parametric form of a plane starting from the vector form.

Example

Represent the plane $(x, y, z) = (8, 5, 3) + s(1, 3, -2) + t(4, -3, 2)$ in scalar form and in parametric form.

Solution

Since you have two direction vectors, you can find a normal vector by taking their cross product:

$(1, 3, -2) \times (4, -3, 2) = (0, -10, -15)$

Therefore, using the normal vector, a scalar equation for the plane will have this form:

$0x - 10y - 15z + d = 0$

Now you have to find the value of d. Do this by substituting the given point $(8, 5, 3)$:

$0(8) - 10(5) - 15(3) + d = 0$

$-50 - 45 + d = 0$

$d = 95$

A scalar equation for the plane is $-10y - 15z + 95 = 0$.

Parametric equations for the plane can be found simply by separating the given vector equation ($(x, y, z) = (8, 5, 3) + s(1, 3, -2) + t(4, -3, 2)$) into one equation for each coordinate:

$x = 8 + s + 4t$

$y = 5 + 3s - 3t$

$z = 3 - 2s + 2t$

Here is another example in which you find the equation of a plane that passes through a point and is parallel to another plane.

Example

Write a scalar equation for the plane that contains the point $(-1, 2, 0)$ and is parallel to the plane $(x, y, z) = (1, -1, 0) + s(1, 1, -1) + t(-1, 2, 1)$.

Solution

If two planes are parallel, they have the same normal vector. You can find a normal vector for the given plane by taking the cross product of the two given direction vectors:

$(1, 1, -1) \times (-1, 2, 1) = (3, 0, 3)$

Therefore, the given plane and any plane parallel to it will have normal vector $(3, 0, 3)$ and scalar equation of this form:

$3x + 0y + 3z + d = 0$

Since the plane in this case must contain the point $(-1, 2, 0)$, you can substitute these coordinates to find the value of d:

$3(-1) + 0(2) + 3(0) + d = 0$

$-3 + 0 + 0 + d = 0$

$d = 3$

A scalar equation of the plane is $3x + 3z + 3 = 0$.

Support Question
(do not send in for evaluation)

22. Write a scalar equation for the plane that contains the point $(1, 0, 2)$ and is parallel to the plane $(x, y, z) = (2, -1, 1) + s(1, 2, -1) + t(0, -2, 1)$.

Conclusion

You are now familiar with the various ways you can represent planes and lines. In the next and final lesson, Lesson 20, you will learn how to solve algebraically for the intersection of planes and lines and interpret your findings geometrically. You will also learn how to solve problems involving distances and projections.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(19 marks)

56. Write a vector equation and a parametric equation for each line: **(6 marks)**
    a) The line through $A(1, -3, 1)$ and parallel to $\vec{u} = (2, -2, 1)$
    b) The line through $A(3, 0, 4)$ and parallel to the x-axis
    c) The line through $A(-1, 2, 1)$ and $B(1, 2, 1)$
57. Determine scalar, vector, and parametric equations for the plane that passes through the points $A(1, -2, 0)$, $B(1, -2, 2)$, and $C(0, 3, 2)$. **(8 marks)**
58. Write a scalar equation for the plane that contains the point $(-1, 2, 0)$ and is parallel to the plane $(x, y, z) = (1, -2, 1) + s(3, 1, 1) + t(4, -2, 1)$. **(5 marks)**

Now go on to Lesson 20. Do not submit your coursework to ILC until you have completed Unit 4 (Lessons 16 to 20).