MCV4U-A

18

Lines and Planes Using Linear Equations

Introduction

Can you describe the difference between a line and a plane?

In this lesson, you will explore linear equations in two-space and three-space. You will also learn about the various ways that lines and planes can intersect in two-space and three-space.

The focus in this lesson is more geometric than algebraic, so you will be making extensive use of a graphing applet to graph various linear equations.

Estimated Hours for Completing This Lesson	
Linear Equations in Two-Space	1
Systems of Linear Equations in Two-Space	1
Linear Equations in Three-Space	1
Systems of Linear Equations in Three-Space	1
Key Questions	1

What You Will Learn

After completing this lesson, you will be able to

- solve systems of linear equations in two-space by graphing
- understand the geometric significance of linear equations in two-space and in three-space
- identify the various ways in which planes can intersect in three-space

Linear Equations

In an equation containing constants and variables, if the variables are not multiplied or raised to a power, the equation is called a linear equation. Here are some examples of linear equations:

- $3x - 10y = 5$
- $5a + 12b - 4c + 8d - 11 = 0$
- $42u - 10.5v - 86.2w + 47.3 = 0$

In this lesson, you will focus mainly on two types of linear equations:

- A linear equation in the two variables x, y in two-space
- A linear equation in the three variables x, y, z in three-space

You're interested in how to interpret these types of equations geometrically. You'll also look at what it means geometrically to have a system of more than one linear equation.

Linear Equations in Two-Space

In two-space, suppose you have a single linear equation with two variables, x and y. Examples of such equations would be $2x - 4y - 5 = 0$ or $3x - 10y + 7 = 0$. Any linear equation in two-space can be written in the form $ax + by + c = 0$, where a, b, and c are constants.

The first key fact of this lesson is that a single linear equation in two-space always represents a straight line. There are a few different ways to demonstrate this, as shown in the following examples.

Example

Consider the linear equation $2x - 4y - 5 = 0$ in two-space. What points (x, y) satisfy this equation? What do the solution points look like when you graph them?

Solution

One approach to visualize the equation $2x - 4y - 5 = 0$ is to solve for y in terms of the other variable:

$$2x - 4y - 5 = 0$$

$$-4y = -2x + 5$$

$$y = \frac{-2}{-4}x + \frac{5}{-4}$$

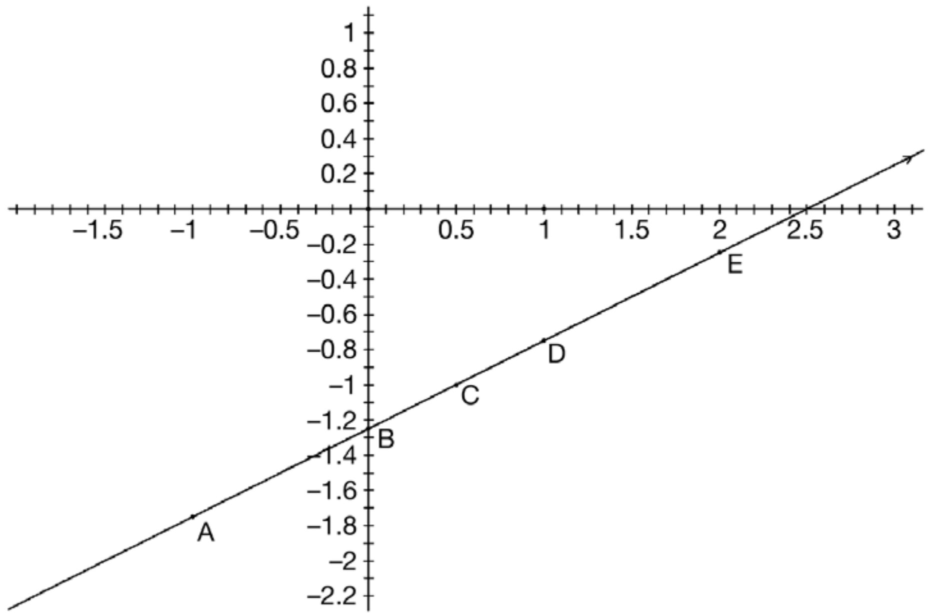
$$y = \frac{1}{2}x - \frac{5}{4}$$

You probably recognize this as the slope-intercept form of a straight line that you've seen in previous courses.

Substitute any value you like for x, and solve for y. Then construct a table of values of x and y and plot the corresponding points on a graph. Note that if you prefer to work with decimals rather than fractions, instead of writing $y = \frac{1}{2}x - \frac{5}{4}$ you can write the equation as $y = 0.5x - 1.25$.

x	$0.5x - 1.25 = y$	Point (x, y)
–1	$0.5(-1) - 1.25 = -1.75$	$(-1, -1.75)$
0	$0.5(0) - 1.25 = -1.25$	$(0, -1.25)$
0.5	$0.5(0.5) - 1.25 = -1.00$	$(0.5, -1.00)$
1	$0.5(1) - 1.25 = -0.75$	$(1, -0.75)$
2	$0.5(2) - 1.25 = -0.25$	$(2, -0.25)$

If you plot these five points on a graph, you can see that they all lie on a straight line, just as expected. (The five points are labelled A through E.)



You can also explore a linear equation in two-space using the graphing applet "Lesson 18 Activity 1" on your course page at *ilc.org*. Open the applet and recreate this graph. Here are the steps:

1. Rearrange the equation into slope-intercept form, just as you did earlier when you rearranged $2x - 4y - 5 = 0$ into $y = \frac{1}{2}x - \frac{5}{4}$.
2. Type the formula, either in the form $\frac{1}{2}x - \frac{5}{4}$ or $0.5x - 1.25$, into the input area and hit Enter.

You can now experiment with creating points on the line, measuring their coordinates, and so on. Be aware that in a linear equation it's possible for a variable to be "missing." This is illustrated by the next example.

Example

Consider the following two equations:

a) $3x - 7 = 0$

b) $5y = -8$

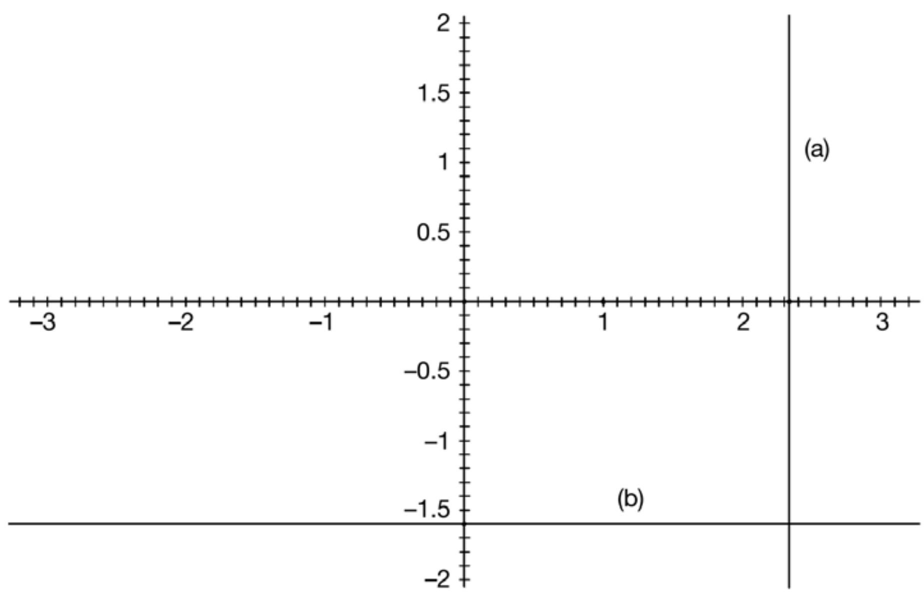
Are these linear equations? What do their graphs look like?

Solution

Both $3x - 7 = 0$ and $5y = -8$ are examples of linear equations, and their graphs are straight lines. Equations (a) and (b) can be rewritten as $3x + 0y - 7 = 0$ and $0x + 5y + 8 = 0$ respectively; that is, both of them can be written in the form $ax + by + c = 0$.

Equation (a) can also be rewritten as $x = \frac{7}{3}$. This means that the graph of (a) consists of all points where $x = \frac{7}{3}$, and y can be any value. Thus, the graph of (a) is a vertical line, as shown in the following graph.

Equation (b) can be rewritten as $y = \frac{-8}{5}$. Similarly, the graph of (b) consists of all points where $y = \frac{-8}{5}$ and x can be any value, so the graph of (b) is a horizontal line as shown in the following graph.



Notice that equation (a) can't be rewritten in slope-intercept form, since the variable y is not in the equation. Equation (a) nevertheless represents a straight line, but its slope is undefined. In contrast, equation (b) can be written in slope-intercept form as $y = 0x - \frac{8}{5}$, so the graph is a straight line with slope 0.

General fact: In two-space, any equation in the form $ax + by + c = 0$, where a, b, and c are constants, is called a linear equation. As long as a and b are not both zero, the graph of such an equation in two-space is a straight line.

If a is zero, the graph is a horizontal line and has slope zero.

If b is zero, the graph is a vertical line and its slope is undefined.

If a and b are both non-zero, the graph is a "typical" straight line that is neither horizontal nor vertical, and its slope is a non-zero value.

If a and b are both zero, the equation contains no variables and there is no graph.

Systems of Linear Equations in Two-Space

A system of equations simply means a collection of several equations. Solving a system of equations means finding those points that satisfy all the equations in the system.

If the equations in the system are linear equations in two-space, then each equation represents a straight line in the plane. The solution to the system consists of those points, if any, that are common to all the straight lines.

In general, a system of linear equations can be solved either geometrically or algebraically. This lesson focuses on the geometric approach.

Example

Solve the system of linear equations in two-space by graphing the lines:

a) $2x + 5y + 15 = 0$

b) $3x - 4y + 11 = 0$

Solution

Rewrite equation (a) in slope-intercept form:

$2x + 5y + 15 = 0$

$5y = -2x - 15$

$y = \dfrac{-2}{5}x - 3$

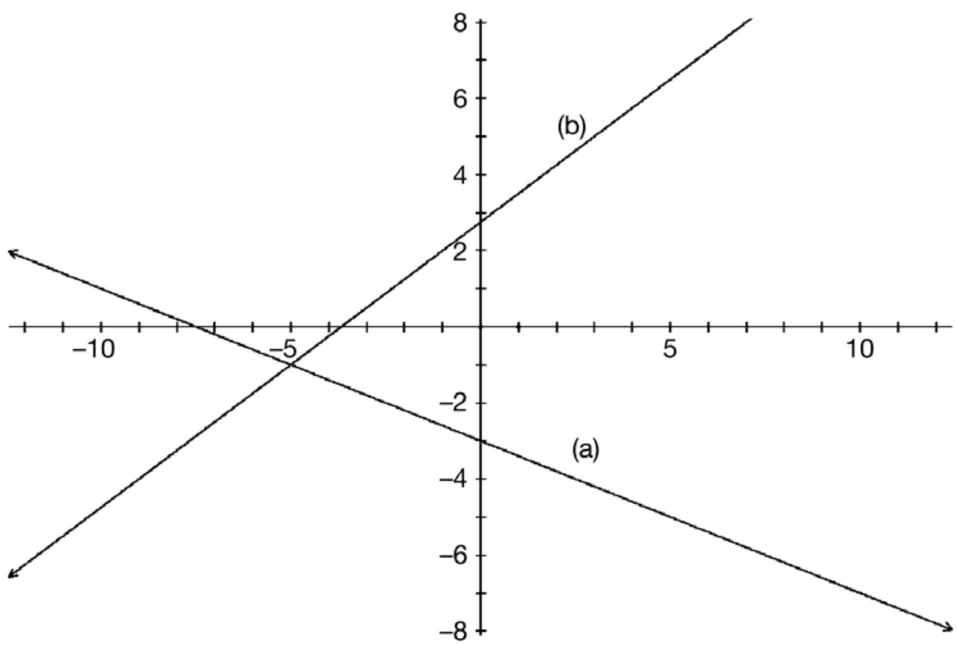
Rewrite equation (b) in slope-intercept form:

$3x - 4y + 11 = 0$

$-4y = -3x - 11$

$y = \dfrac{3}{4}x + \dfrac{11}{4}$

Slope-intercept form makes it easier to plot both lines on the same graph:



From the graph, it looks as if the two lines intersect around the point (–5, –1). If you substitute these values into equations (a) and (b), you find that both equations are satisfied. Not surprisingly, if your goal is just to find the exact coordinates of the point where two lines intersect, the quickest and most direct way is to approach the problem algebraically. The focus in this lesson, however, is on the geometric interpretation of systems of linear equations.

Example

Solve the system of linear equations in two-space by graphing the lines:

a) $2x + 5y + 15 = 0$

b) $-2x - 5y + 11 = 0$

Solution

As seen in the previous example, equation (a) in slope-intercept form is

$$y = \frac{-2}{5}x - 3$$

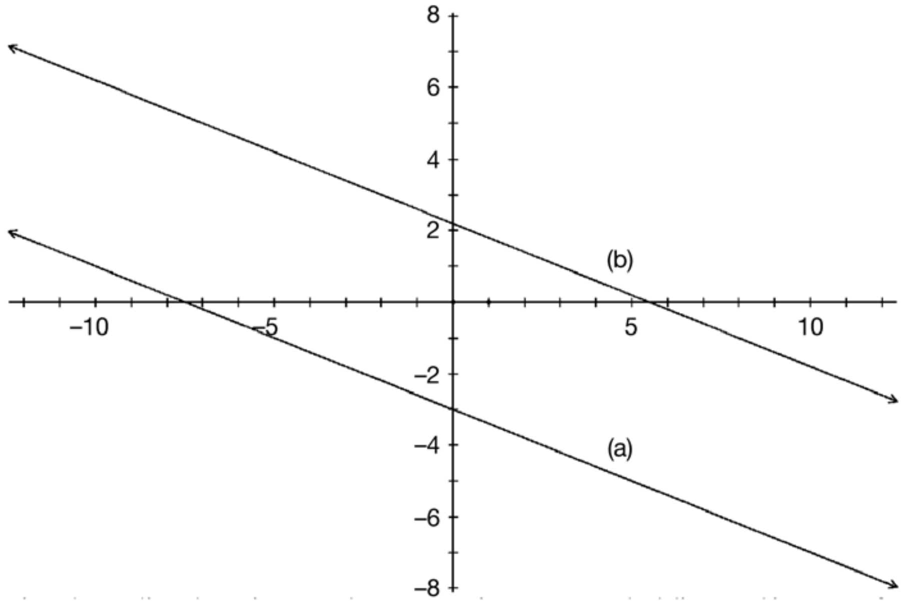
Rewrite equation (b) in slope-intercept form:

$$-2x - 5y + 11 = 0$$

$$-5y = 2x - 11$$

$$y = \frac{-2}{5}x + \frac{11}{5}$$

You can now plot the two lines on the same graph. In this case, the two lines have the same slope but different y-intercepts, so they are two parallel lines.



Since the two lines do not intersect, there are no points common to both lines, so this system of linear equations has no solution.

Example

Solve the system of linear equations in two-space by graphing the lines:

a) $2x + 5y + 15 = 0$

b) $4x + 10y + 30 = 0$

Solution

Again, equation (a) in slope-intercept form is

$$y = \frac{-2}{5}x - 3$$

Rewrite equation (b) in slope-intercept form:

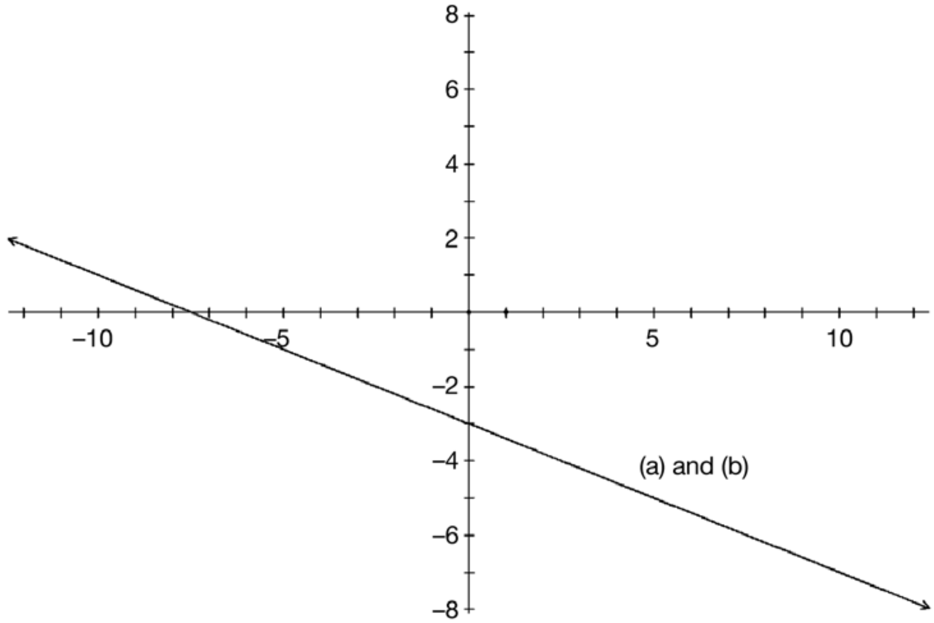
$$4x + 10y + 30 = 0$$

$$10y = -4x - 30$$

$$y = \frac{-4}{10}x - \frac{30}{10}$$

$$y = \frac{-2}{5}x - 3$$

Even though equations (a) and (b) initially appeared different, you found after rewriting them in slope-intercept form that they actually represent the same line, as shown in the following graph.



These two lines are coincident, meaning that they are the same line. They intersect in an infinite number of points because any point on one of the lines is also on the other. Thus, this system of linear equations has an infinite number of solutions.

If you imagine having two straight lines in the plane and experiment with the different possible ways they could intersect, you will discover that the three examples cover all the possibilities. Two lines in a plane can intersect in a point, be parallel, or be coincident. The corresponding system of linear equations has one solution, no solutions, or an infinite number of solutions.

Support Question
(do not send in for evaluation)

16. Using graph paper, solve the system of linear equations $4x + 2y - 8 = 0$ and $-4x + y + 5 = 0$.

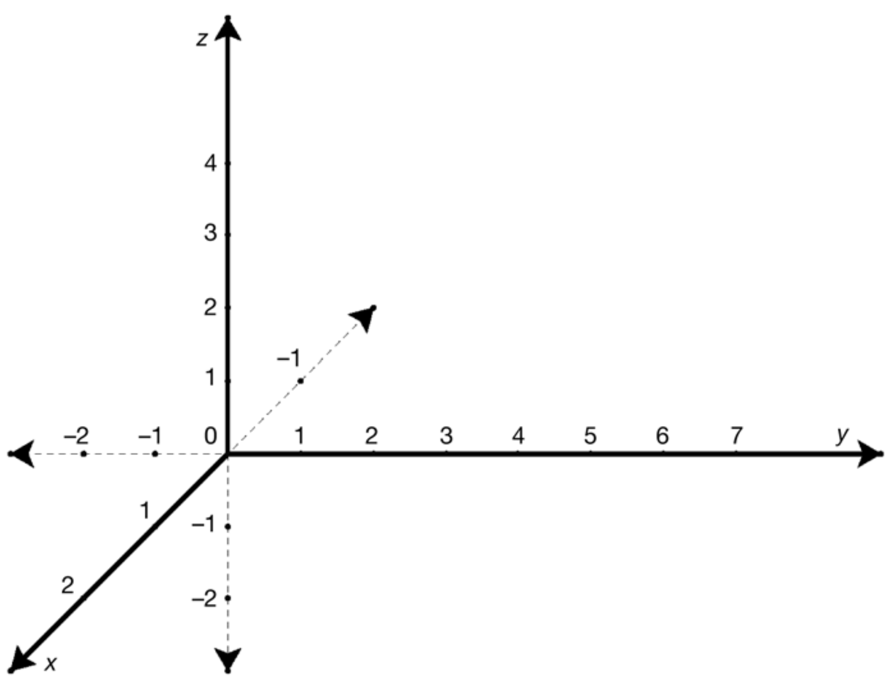
There are Suggested Answers to Support Questions at the end of this unit.

Linear Equations in Three-Space

You've seen in two-space that a linear equation has the form $ax + by + c = 0$ and represents a straight line. If you're working in three-space, then a "linear equation" means any equation of the form $ax + by + cz + d = 0$, where a, b, c, and d are constants. An equation of this form in three-space represents not a straight line but a plane. A plane is an infinite flat surface with no thickness.

Admittedly, three dimensions are harder to visualize than two. Nevertheless, experimenting with specific examples will help illustrate why a single linear equation in three-space represents a plane.

Start by reviewing how to construct a three-space graph. Recall that in three-space you have three perpendicular axes: an x-axis, a y-axis, and a z-axis. Typically, the plane containing the x-axis and y-axis is thought of as horizontal. For example, imagine that the x-axis and y-axis are two perpendicular lines drawn on the surface of your table or desk, and the z-axis is a third line perpendicular to the surface of the desk. The following graph illustrates this:



Example

In three-space, consider the linear equation $x + 3y - z - 2 = 0$. By experimenting with different values of the variables, convince yourself that this equation represents a plane in three-dimensional space, not a line.

Solution

Just as with a linear equation with two variables, it is possible to solve for one of the variables in terms of the others. For example, you can solve for z in terms of x and y:

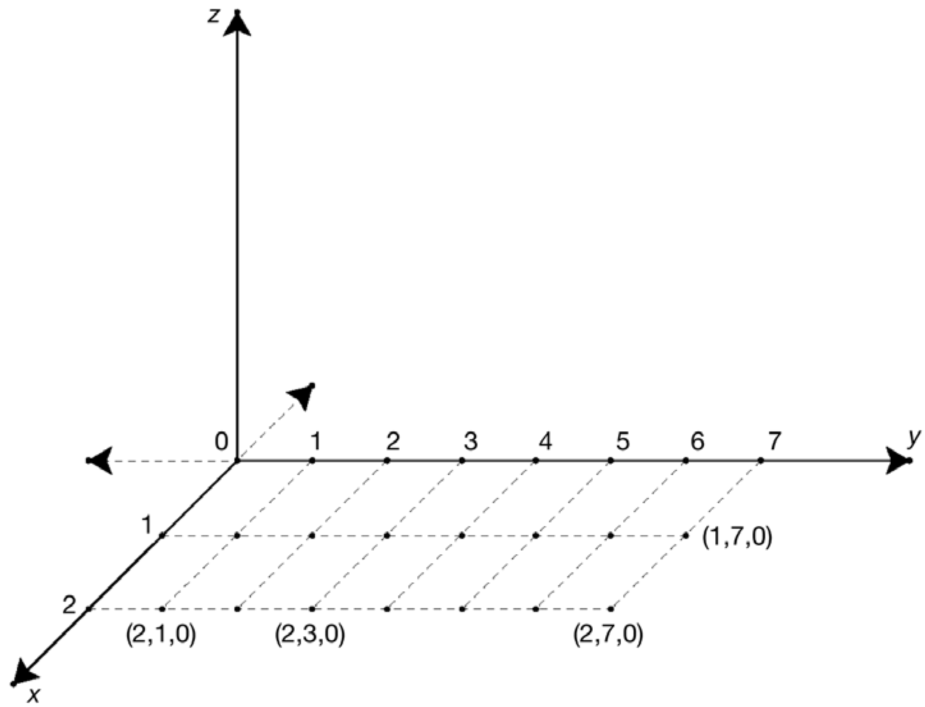
$x + 3y - z - 2 = 0$

$-z = -x - 3y + 2$

$z = x + 3y - 2$

With linear equations in two-space, you solve for y in terms of x, which means that you can arbitrarily use any value for x and solve for y. When you plot all the points generated by that process, you get a straight line.

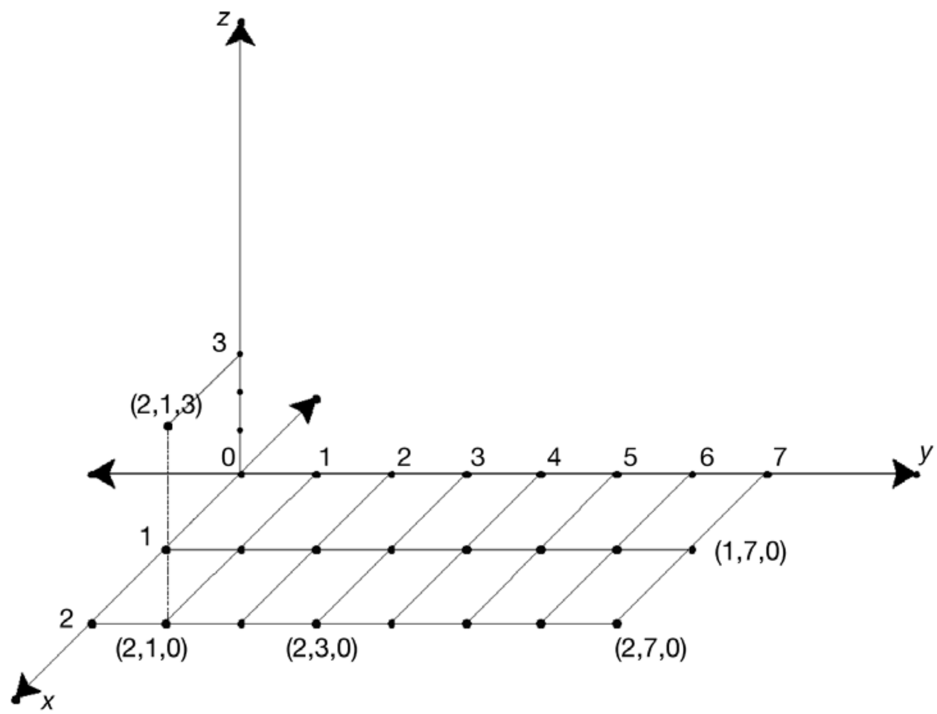
With the linear equation in three-space in the previous example, you can arbitrarily use any values for x and y, and solve for z. To visualize this process, imagine that the surface of your desk or table is the x-y plane; that is, the plane containing those points whose z-coordinate is zero. Points with a positive z-coordinate are located above the surface of the desk and points with a negative z-coordinate are located below the surface of the desk.



For illustration, a few different values of x and y are labelled in the graph. See what happens when you use those values in the equation $z = x + 3y - 2$. The results can be summarized in a table of values:

(x, y)	**$x + 3y - 2 = z$**	**z**
(2, 1)	$2 + 3 \times 1 - 2 = 3$	3
(2, 3)	$2 + 3 \times 3 - 2 = 9$	9
(2, 7)	$2 + 3 \times 7 - 2 = 21$	21
(1, 7)	$1 + 3 \times 7 - 2 = 20$	20

When $x = 2$ and $y = 1$, you get $z = 3$. Thus, the point (2, 1, 3) is one of the points on the graph, and this point is located three units above the point (2, 1, 0). Similarly, when $x = 2$ and $y = 3$, you get $z = 9$, so the point (2, 3, 9) is another point on the graph. This point is located 9 units above the point (2, 3, 0) on the graph. Any value for x and y of zero or less results in a negative value of z, which corresponds to a point below the x-y plane.



You've seen that when you solve for z in terms of x and y, then above or below *every* point in the x-y plane there is a point of the graph. This means that the graph must be some kind of two-dimensional surface. Plotting more points should help to convince you that the graph is a flat plane.

What happens if you keep x constant but vary y? What happens if you keep y constant but vary x?

When you studied linear equations in two-space, you saw that it was possible for one of the variables to be "missing." The same can happen in three-space.

Examples

Here are examples of equations in three-space. Are they linear equations? What do their graphs look like?

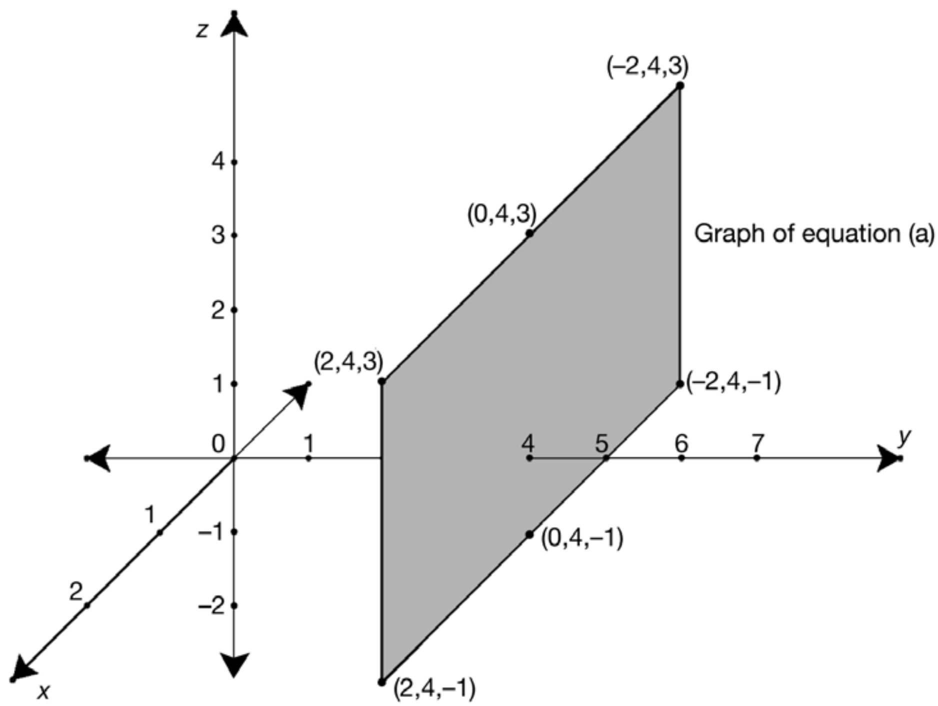
a) $y = 4$

b) $y = 3x$

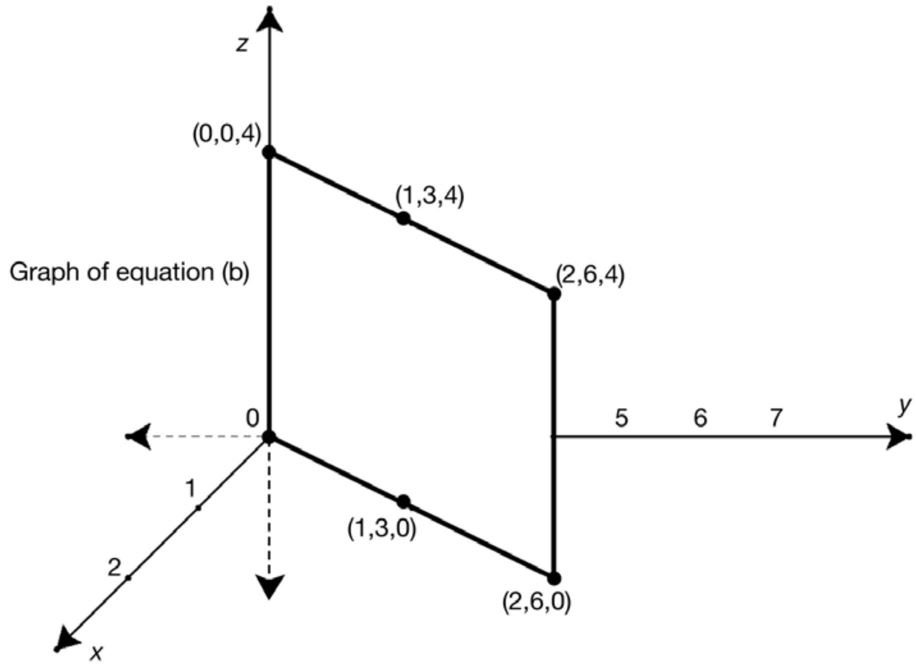
Solutions

Both of these are linear equations because they both can be written in the form $ax + by + cz + d = 0$. Equation (a) can be written as $0x + 1y + 0z - 4 = 0$, and equation (b) can be written as $-3x + 1y + 0z + 0 = 0$.

The graph of equation (a) consists of all points in three-space whose y-coordinate is 4. The x- and z-coordinates can have any value and the equation will be satisfied. Therefore, the graph is a plane parallel to the x-z plane, but located 4 units away from the origin in the positive direction along the y-axis. This is shown in the following graph.



The graph of equation (b), on the other hand, consists of all points in three-space whose coordinates satisfy $y = 3x$. You know that if you restrict your attention to just the x-y plane, the graph of $y = 3x$ is a straight line. If you consider the same equation in three-space, then z can have any value. The graph consists of an infinite number of copies of the line $y = 3x$ located above or below the x-y plane, depending on the value of z. These copies of the line $y = 3x$ form a plane, as shown in the following graph.



General fact: In three-space, any equation of the form $ax + by + cz + d = 0$, where a, b, c, and d are constants, is called a **linear equation**. As long as a, b, and c are not all zero, the graph of such an equation in three-space will form a **plane**.

If a, b, and c are all zero, the equation contains no variables and there is no graph.

Systems of Linear Equations in Three-Space

Just as in two-space, a system of equations means a collection of equations that you're trying to solve. If you have a system of linear equations in three-space, then each equation in the system represents a plane. The solution to the system consists of those points, if any, that lie in all of the planes.

Here is an example of a system of linear equations in three-space:

$x - 3y - 2z = -9$

$2x - 5y + z = 3$

$-3x + 6y + 2z = 8$

Here is another example:

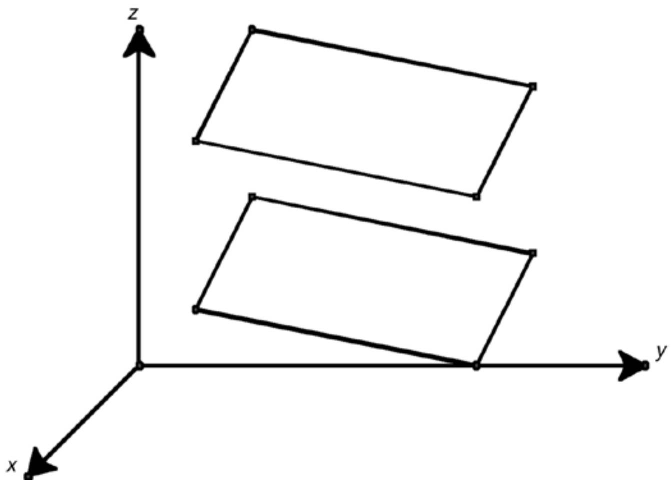
$y = 4$

$y = 3x$

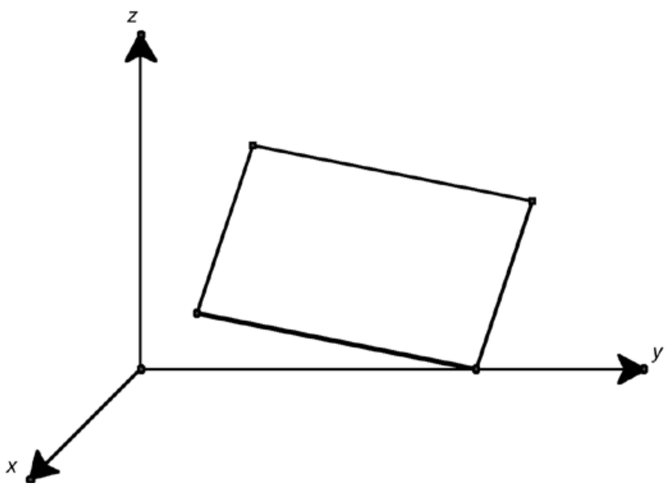
There are algebraic techniques for solving systems of linear equations with three or more variables, but this lesson focuses on the geometric interpretation.

If you have a system of linear equations in three-space, then each equation in the system represents a plane. Two equations means two planes, three equations means three planes, and so on.

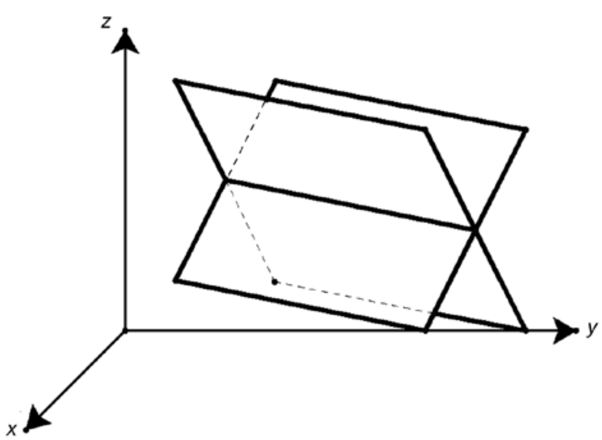
In what possible ways can two planes be situated in three-space? Imagine the two planes are two pieces of paper and experiment with the different ways that you can hold two flat pieces of paper relative to one another. You will discover that there are three possibilities: the planes can be parallel, they can be coincident, or they can intersect in a line. The diagrams illustrate each possibility:



Parallel Planes



Coincident Planes



Planes That Intersect in a Line

There are many possible ways that three planes can be situated in three-space, but first imagine starting with just two of the planes before considering the third one. The first two planes are parallel, coincident, or intersect in a line. Suppose they intersect in a line. If that happens, you then have to consider the various ways in which that line could intersect the third plane.

In what ways can a line and a plane be situated in three-space? Suppose you have a pen, representing a line, and a piece of paper, representing a plane. By experimenting with these objects, you discover that there are three possibilities. The line can lie in the plane, the line can intersect the plane at a point, or the line can be parallel to the plane.

In this way, you can see that the intersection of three planes in three-space could be a line, a point, or a plane. An additional possibility is that the line doesn't intersect the plane at all.

Support Questions
(do not send in for evaluation)

17. Describe the shape of the intersection of the plane $z = 3$ and the plane $y = 2x$ in three-space.

18. Draw diagrams and list all the possibilities for the intersection of three planes in three-space.

Conclusion

In this lesson, you used geometry to look at the way lines and planes can intersect. In Lessons 19 and 20, you will explore the same ideas from an algebraic angle and solve for those intersections.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(14 marks)

53. In two-space, find the intersection of the line $2x + 5y = 18$ and the line $3x + 4y = 20$ by graphing the two lines on graphing paper. **(4 marks)**
54. Describe the shape of the intersection of the plane $z = -3$ and the plane $y = z$ in three-space. **(4 marks)**
55. List all the possibilities for the intersection of a line and a plane, and draw an example of each. **(6 marks)**

Now go on to Lesson 19. Do not submit your coursework to ILC until you have completed Unit 4 (Lessons 16 to 20).

