MCV4U-A

17

Properties and Applications of Dot and Cross Products

Introduction

Is it easier to loosen a bolt holding a wrench in the middle of the handle or at the end? Vector multiplication can help you determine the amount of torque applied when unfastening a bolt.

In Lesson 16, you learned about two types of vector multiplication, the dot product and the cross product. You learned that the dot product of two vectors is a scalar and the cross product of two vectors is a vector.

In this lesson, you will explore properties of the cross product. In addition, you will learn to solve real-world problems using the dot product and cross product.

Estimated Hours for Completing This Lesson	
Properties of Cross Products	1
Dot Product, Cross Product, and Geometry	1.5
Dot Product, Cross Product, and Physics	1.5
Key Questions	1

What You Will Learn

After completing this lesson, you will be able to

- identify properties of the cross product
- solve problems in geometry and physics involving the cross and dot product

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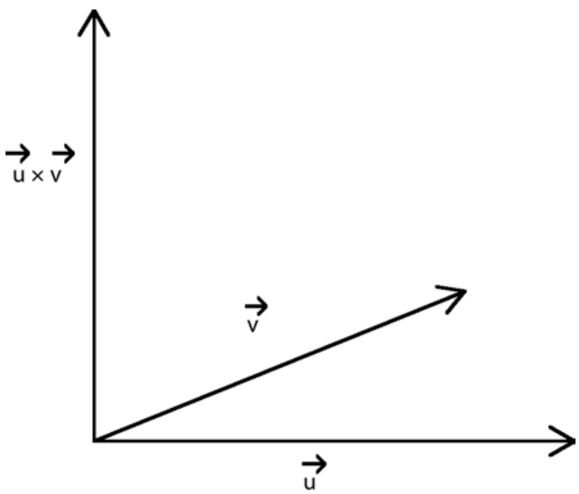
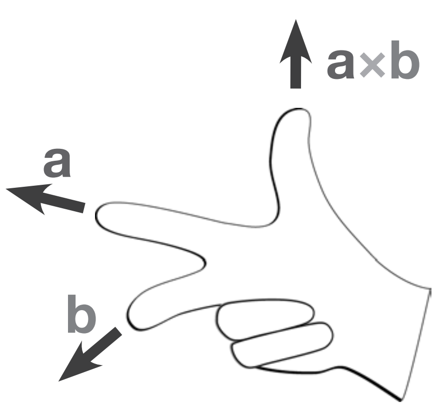
Properties of Cross Products

You studied the cross product of two vectors in Lesson 16 and saw that the result is a vector perpendicular to the plane that contains both vectors.



Right-Hand Rule

The direction of the cross product of two vectors must satisfy the right-hand rule. The right-hand rule states that if the first finger of the right hand is pointing in the direction of $\vec{a}$ and the second finger is pointing toward $\vec{b}$, then the thumb is in the direction of $\vec{a} \times \vec{b}$.



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Magnitude of Cross Product

The magnitude of the cross product of two vectors is $|\vec{u} \times \vec{v}| = |\vec{u}\,||\vec{v}|\sin\theta$ where θ is the angle between the two vectors and $0 \le \theta \le 180$.

Example

Find the magnitude of $\vec{u} \times \vec{v}$ if $|\vec{u}\,| = 14$, $|\vec{v}| = 10$, and the angle between them is 60°.

Solution

$$\begin{aligned} |\vec{u} \times \vec{v}| &= |\vec{u}\,||\vec{v}|\sin\theta \\ &= (14)(10)\sin(60) \\ &\approx 121.24 \end{aligned}$$

Commutative Property

In Lesson 16, one of the properties you looked at that related to the dot product was the commutative property. Does the commutative property hold for the cross product?

Find and compare $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$ for $\vec{u} = (1, 0, -1)$ and $\vec{v} = (2, -1, -1)$.

Using the cross product procedure outlined in Lesson 16, calculate the following results:

$\vec{u} \times \vec{v} = (-1, -1, -1)$ and $\vec{v} \times \vec{u} = (1, 1, 1)$

What do you notice?

$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

This property is referred to as the anti-commutative law: when the order is changed, the result is the negative of the original result.

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Distributive Property

Similar to the dot product, the distributive property holds for the cross product. That is:

$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

Here's an example to illustrate this property. Note that in general an example is not sufficient to show that a property holds.

Example

Show that the distributive property holds for the following vectors:

$\vec{u} = (1, -1, 2)$, $\vec{v} = (1, 0, 2)$, and $\vec{w} = (-1, -3, 1)$

Solution

$\vec{u} \times (\vec{v} + \vec{w})$:

$(\vec{v} + \vec{w}) = (0, -3, 3)$

$$\begin{array}{cccc} -1 & 2 & 1 & -1 \\ -3 & 3 & 0 & -3 \end{array}$$

$(-1)(3) - (-3)(2) = 3$

$(2)(0) - (3)(1) = -3$

$(1)(-3) - (0)(-1) = -3$

$\vec{u} \times (\vec{v} + \vec{w}) = (3, -3, -3)$

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Next, use the process for the cross product to find the following:

$$\vec{u} \times \vec{v} + \vec{u} \times \vec{w} = (-2, 0, 1) + (5, -3, -4)$$

$$= (3, -3, -3)$$

As a result, you can conclude that the distributive property holds for the cross product.

Associative Property

Does the associative property hold for the cross product? The associative property deals with the cross product of three vectors and would be defined as follows:

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$$

The following example illustrates that this property does not hold for cross product.

Example

Show that the associative property does not hold by using the following vectors:

$\vec{u} = (1, -1, 2)$, $\vec{v} = (1, 0, 2)$, and $\vec{w} = (-1, -3, 1)$

Solution

To determine if $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$, start by calculating the left-hand side of the equation:

$$\vec{u} \times (\vec{v} \times \vec{w})$$

$$\vec{v} \times \vec{w} = (6, -3, -3)$$

$$\vec{u} \times (\vec{v} \times \vec{w}) = (1, -1, 2) \times (6, -3, -3)$$

$$= (9, 15, 3)$$

Then calculate the right-hand side of the equation:

$$\vec{u} \times \vec{v} = (-2, 0, 1)$$

$$(\vec{u} \times \vec{v}) \times \vec{w} = (-2, 0, 1) \times (-1, -3, 1)$$

$$= (3, 1, 6)$$

You can conclude that $\vec{u} \times (\vec{v} \times \vec{w}) \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

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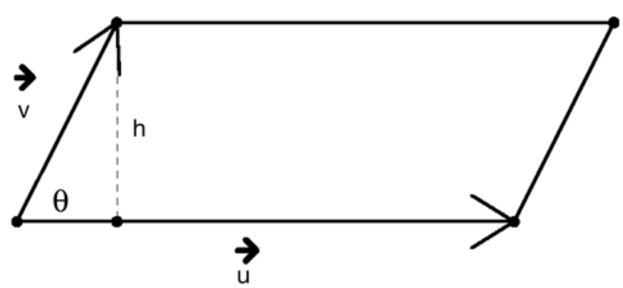
Dot Product, Cross Product, and Geometry

The dot product and cross product of vectors can be applied to solve problems in geometry, specifically problems related to area, volume, and projection.

Area of a Parallelogram

Recall that the area of a parallelogram is its base times its height: $A = bh$.

In the following diagram, the base is equal to the magnitude of $\vec{u}$, and h is calculated by using the sine ratio of the angle θ, which is $h = |\vec{v}|\sin\theta$. If $A = bh$, then the area of the parallelogram with sides $\vec{u}$ and $\vec{v}$ is $A = |\vec{u}||\vec{v}|\sin\theta$. This is the same as the magnitude of the cross product of $\vec{u}$ and $\vec{v}$.



> The area of a parallelogram with sides $\vec{u}$ and $\vec{v}$ is $A = |\vec{u} \times \vec{v}|$.

Here's an example to show how this formula can be used to calculate the area of a triangle.

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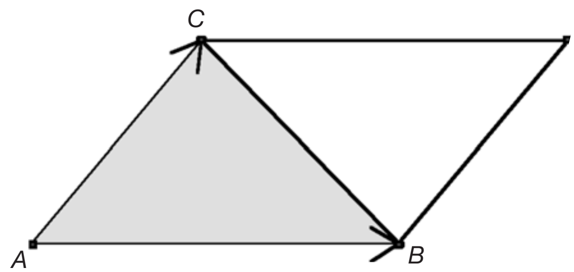
Example

Find the area of the triangle with vertices $A(-1, 2, 5)$, $B(2, 1, 2)$, and $C(2, 3, 4)$.

Solution

The area of the triangle ABC is half the area of the parallelogram determined by the vectors $\vec{AB}$ and $\vec{AC}$:



Calculate the area of the parallelogram using the magnitude of the cross product of $\vec{AB}$ and $\vec{AC}$.

Determine the coordinates of the vectors:

$$\vec{AB} = (2 - (-1), 1 - 2, 2 - 5) \qquad \vec{AC} = (2 - (-1), 3 - 2, 4 - 5)$$

$$= (3, -1, -3) \qquad = (3, 1, -1)$$

Find the cross product $\vec{AB} \times \vec{AC}$,

$$\begin{array}{cccc} -1 & -3 & 3 & -1 \\ 1 & -1 & 3 & 1 \end{array}$$

$$(-1)(-1) - (1)(-3) = 4$$

$$(-3)(3) - (-1)(3) = -6$$

$$(3)(1) - (3)(-1) = 6$$

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$\vec{AB} \times \vec{AC} = (4, -6, 6)$

The area of the parallelogram is the magnitude of $\vec{AB} \times \vec{AC}$.

$|\vec{AB} \times \vec{AC}| = \sqrt{(4)^2 + (-6)^2 + (6)^2}$

$= \sqrt{88}$

$= 2\sqrt{22}$

The area of the triangle ABC is the area of the parallelogram divided by 2. Therefore, the area of the triangle ABC is $\sqrt{22}$ square units.

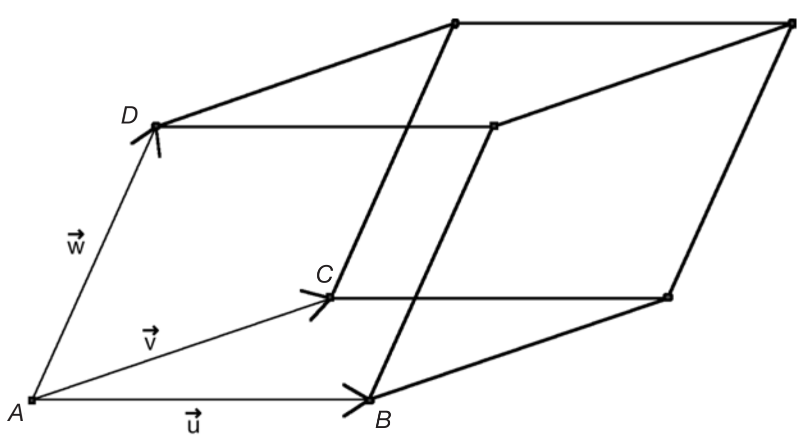
Support Questions
(do not send in for evaluation)

9. Calculate the area of the parallelogram with sides defined by the vectors (1, 2, 4) and (–1, 3, –4).
10. Calculate the area of the triangle ABC with vertices $A(-1, 0, 1)$, $B(-3, 1, 0)$, and $C(-2, -1, 2)$.

There are Suggested Answers to Support Questions at the end of this unit.

Volume of a Parallelepiped

A parallelepiped is a three-dimensional prism where each side is a parallelogram. Its volume is calculated using cross product and dot product.



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The volume of a parallelepiped defined by three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is given by $V = |(\vec{u} \times \vec{v}) \cdot \vec{w}|$.

Example

Find the volume of the parallelepiped defined by the vectors $\vec{u} = (2, -3, -1)$, $\vec{v} = (-1, 0, 2)$, and $\vec{w} = (3, -1, 2)$.

Solution

$$\begin{aligned} V &= |(\vec{u} \times \vec{v}) \cdot \vec{w}| \\ &= |((2, -3, -1) \times (-1, 0, 2)) \cdot (3, -1, 2)| \end{aligned}$$

Use the usual method to calculate the cross product of $\vec{u}$ and $\vec{v}$ to get $(-6, -3, -3)$.

To calculate the volume, find the dot product of $(-6, -3, -3)$ and $(3, -1, 2)$.

$$\begin{aligned} V &= \left|(\vec{u} \times \vec{v}) \cdot \vec{w}\right| \\ &= \left|(-6,\ -3,\ -3) \cdot (3,\ -1, 2)\right| \\ &= \left|(-6)(3) + (-3)(-1) + (-3)(2)\right| \\ &= |-21| \\ &= 21 \end{aligned}$$

The volume of the parallelepiped is 21 cubic units.

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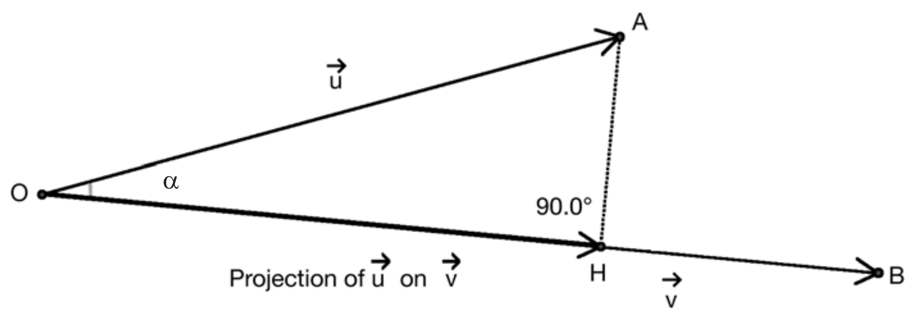
Projection of Vectors

Given two vectors $\vec{u}$ and $\vec{v}$, the projection of $\vec{u}$ on $\vec{v}$ is $\vec{OH}$ where H is the intersection of the perpendicular line from the tip of vector $\vec{u}$ on $\vec{v}$.

You can think of the projection of $\vec{u}$ on $\vec{v}$ as the shadow of $\vec{u}$ on $\vec{v}$ caused by a light shining perpendicularly on $\vec{v}$.



Write the projection of $\vec{u}$ on $\vec{v}$ as $proj_{\vec{v}}\,\vec{u}$ and find it using this formula:

$$proj_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

Here's an explanation of where this formula comes from:

$proj_{\vec{v}}\,\vec{u}$, which is equal to $\vec{OH}$, is a scalar multiple of $\vec{v}$ since it has the same direction as $\vec{v}$. So, $\vec{OH} = k\vec{v}$, where k is a real number.

The vectors $\vec{v}$ and $\vec{OH}$ have the same direction, so the vector obtained from $\vec{v}$ divided by $|\vec{v}|$ has a magnitude equal to 1 and has the same direction as $\vec{OH}$.

The vector $|\vec{OH}|\dfrac{\vec{v}}{|\vec{v}|}$ has the same direction as $\vec{OH}$ and the same magnitude.

So, $\vec{OH} = |\vec{OH}|\dfrac{\vec{v}}{|\vec{v}|}$ **[1]**

Use simple trigonometry to find the magnitude of $|\vec{OH}|$ in terms of $|\vec{u}|$ and $|\vec{v}|$.

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Use the cosine ratio:

$$\cos\alpha = \frac{|\vec{OH}|}{|\vec{u}|}$$

$$\left|\vec{OH}\right| = |\vec{u}|\cos\alpha$$

Given the definition of the dot product, you know that $\vec{u}\cdot\vec{v} = \left|\vec{u}\right|\left|\vec{v}\right|\cos\alpha$.

It follows that $\cos(\alpha) = \dfrac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}$. Substitute this value back into $|\vec{OH}|$:

$$\begin{aligned}\left|\vec{OH}\right| &= |\vec{u}|\cos\alpha \\ &= |\vec{u}|\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} \\ &= \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\end{aligned}$$

Substitute $\left|\vec{OH}\right| = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$ into [1].

$$\begin{aligned}\vec{OH} &= \left|\vec{OH}\right|\frac{\vec{v}}{|\vec{v}|} \\ &= \frac{\left(\vec{u}\cdot\vec{v}\right)}{|\vec{v}|}\frac{\vec{v}}{|\vec{v}|} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}||\vec{v}|}\right)\vec{v} \\ &= \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}\end{aligned}$$

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Therefore, $proj_{\vec{v}}\,\vec{u} = \left(\dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)\vec{v}$

Example

Find the projection of $\vec{u} = (2, 3, 4)$ onto $\vec{v} = (2, 1, -3)$.

Solution

$$\begin{aligned}
proj_{\vec{v}}\,\vec{u} &= \left(\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\right)\vec{v} \\
&= \frac{(2, 3, 4) \cdot (2, 1, -3)}{(2, 1, -3) \cdot (2, 1, -3)}(2, 1, -3) \\
&= \frac{4 + 3 - 12}{4 + 1 + 9}(2, 1, -3) \\
&= \frac{-5}{14}(2, 1, -3) \\
&= \left(\frac{-5}{7}, \frac{-5}{14}, \frac{15}{14}\right)
\end{aligned}$$

Support Questions
(do not send in for evaluation)

11. Find the volume of the parallelepiped defined by the vectors (1, –1, 0), (1, 0, 3), and (0, –1, 1).
12. Find the projection of $\vec{a} = (4, -1, 4)$ on $\vec{b} = (-2, 0, 4)$.

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Dot Product, Cross Product, and Physics

In physics, there are a number of applications for the dot product and cross product of vectors, such as when calculating work and torque.

Work

Work, in physics, happens when a force acting on an object causes displacement, such as the work done when moving a desk. Work done by a force is defined by a dot product:

$$\begin{aligned} W &= \vec{F} \cdot \vec{d} \\ &= |\vec{F}||\vec{d}|\cos\theta \end{aligned}$$

Where $\vec{F}$ is the force acting on an object, $\vec{d}$ is the displacement caused by the force. θ is the angle between the force and the displacement vector.

Work is a scalar quantity and its unit is joules (J). The magnitude of the force should be in newtons (N) and the displacement in metres.

Example

A cart is pushed 6 m by a 20 N force acting at a 15° angle to the path of the cart.

Solution

$$\begin{aligned} W &= \vec{F} \cdot \vec{d} \\ &= |\vec{F}||\vec{d}|\cos\theta \\ &= (20)(6)\cos(15) \\ &\approx 115.91 \end{aligned}$$

The work done is approximately 115.91 J.

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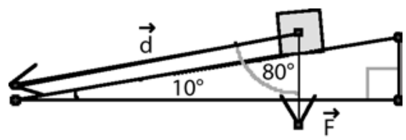
Example

A 20 kg carton is placed at the top of a 10 m ramp that has an incline of 10°.

Find the work done by gravity as the carton slides down to the bottom of the ramp.



Recall that acceleration due to gravity is equal to 9.81 m/s^2. The magnitude of the force caused by gravity is (20)(9.81) = 196.2 N.

$$\begin{aligned} W &= |\vec{F}|\,|\vec{d}|\cos(\theta) \\ &= (196.2)(10)\cos(80) \\ &= 340.7 \end{aligned}$$

The work done by gravity is approximately 340.7 J.

Torque

Suppose you are tightening a bolt with a wrench. Does it make a difference where you grab the handle? Is it easier to tighten the bolt if you hold the handle in the middle?

In physics, torque is a vector quantity that helps to measure the tendency of a force to rotate an object. The torque vector, $\vec{\tau}$, is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}|\,|\vec{F}|\sin\theta$$

$\vec{F}$ is the force applied and $\vec{r}$ is the radius of the rotation measured from the object to the point where the force is applied. Torque is measured in newton-metres (Nm). The magnitude of the force is measured in newtons and the radius is measured in metres.

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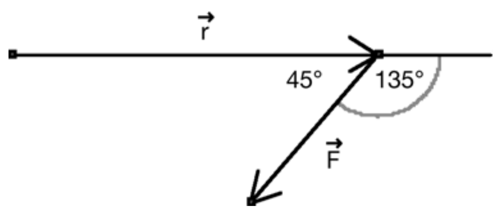
Example

a) Find the torque produced when a 40 N force is applied at a 45° angle to the horizontal at the end of a 30 cm wrench.

b) What is the maximum torque that can be applied by the 40 N force on the wrench and how can this be achieved?

Solution



The torque vector, $\vec{\tau}$, is defined as

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$

The wrench is 30 cm long, so $|\vec{r}| = 0.3$.

The vectors must be aligned tail to tail:

$$\begin{aligned}\theta &= 180 - 45 \\ &= 135\end{aligned}$$

You are now ready to calculate the torque:

$$\begin{aligned}|\vec{\tau}| &= |\vec{r}||\vec{F}|\sin\theta \\ &= (0.3)(40)\sin(135) \\ &\approx 8.48\end{aligned}$$

The torque is approximately 8.48 Nm.

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To calculate the maximum torque, use the largest value of the sine function, which is 1 when $\theta = 90°$.

The maximum torque is

$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$

$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin 90$

$= (0.3)(40)(1)$

$= 12$

The maximum torque that can be applied to the wrench is 12 Nm, and this can be achieved when the force is applied at a 90° angle.

Support Questions
(do not send in for evaluation)

13. Find the work done pulling a wagon 60 m with a 40 N force if the handle makes an angle of 30° with the ground.

14. a) Find the torque produced when a 70 N force is applied at a 35° angle to the horizontal at the end of a 40 cm wrench.

    b) What is the maximum torque that can be applied by the 70 N force on the wrench and how can this be achieved?

15. Find the torque produced by a cyclist exerting a force of 100 N on a pedal that is 18 cm long if the angle between the force and the pedal is 105°.

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Conclusion

In this lesson, you looked at properties and applications of the dot and cross products. In Lesson 18, you will explore lines and planes and the way they intersect.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(13 marks)

50. Calculate the area of the triangle ABC where A is $(-1, 2, 4)$, B is $(1, -1, 2)$, and C is $(2, 3, 4)$. **(4 marks)**

51. Calculate the volume of the parallelepiped defined by the vectors $\vec{a} = (2, -5, -1)$, $\vec{b} = (2, 0, -1)$, $\vec{c} = (-1, 0, 1)$. **(4 marks)**

52. a) Find the torque produced when a 50 N force is applied at a 30° angle to the horizontal at the end of a 30 cm wrench. What is the maximum torque that can be applied by the 50 N force on the wrench and how can this be achieved? **(3 marks)**

    b) While moving a couch 5 m across the room, you have to lift and drag it at the same time. You exert a force of 200 N at an angle of 70° to the floor as you drag the couch. How much work do you do moving the couch? **(2 marks)**

Now go on to Lesson 18. Do not submit your coursework to ILC until you have completed Unit 4 (Lessons 16 to 20).

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