MCV4U-A

15

Properties of Vectors and Scalar Multiplication

Introduction

In Lesson 14, you learned about vectors in three-space, and adding and subtracting vectors. In this lesson, you will learn about scalar multiplication, and some of the properties or rules for manipulating vectors. You will also use this knowledge to answer some real-world problems involving vectors.

What type of real-world problems? As an example, you can use vectors to determine which direction a pilot should steer an airplane if you know the distance to the destination airport as well as the direction and velocity of the wind and of the airplane.

Estimated Hours for Completing This Lesson	
Multiplying a Vector by a Scalar	0.5
Properties of Vector Arithmetic	1
Solving Real-World Problems Using Vectors	2
Key Questions	1.5

What You Will Learn

After completing this lesson, you will be able to

- add and subtract scalar multiples of vectors in two-space and three-space
- apply properties of vector arithmetic such as the associative law and distributive law
- solve real-world problems involving vector addition, vector subtraction, and scalar multiplication

Multiplying a Vector by a Scalar

In earlier lessons, you've seen that vectors can be represented as directed line segments (geometric vectors), and they can be represented in Cartesian form using coordinates. You have learned how to add and subtract vectors in both of those forms.

Not only can you add two vectors or subtract two vectors, it's also possible to multiply a vector by a scalar. If $\vec{v}$ is any vector and k is any scalar, then $k\vec{v}$ is a new vector with a length that is k times the length of $\vec{v}$. Vectors that are scalar multiples of $\vec{v}$ are described as parallel to $\vec{v}$, although the magnitude may change depending on the value of k. If k is negative, then $k\vec{v}$ will be parallel to $\vec{v}$, but it will face in the opposite direction to $\vec{v}$.

Consider some examples, using both geometric vectors and Cartesian vectors.

Scalar Multiplication of Geometric Vectors

If $\vec{v}$ is a vector represented by a directed line segment, then $k\vec{v}$ is a vector parallel to $\vec{v}$, whose magnitude and direction has been modified appropriately. If $\vec{v}$ is a vector quantity such as a velocity or a force, it is physically meaningful to talk about multiplying $\vec{v}$ by a positive or negative number.

Examples

A hot-air balloon is travelling N10°E at a speed of 30 km/h. Let $\vec{v}$ be the vector representing the balloon's velocity. Find the magnitude and direction of each of the following vectors:

a) $2\vec{v}$

b) $0.8\vec{v}$

c) $-\vec{v}$

d) $-\frac{1}{2}\vec{v}$

Solutions

a) Vector $2\vec{v}$ has magnitude 60 km/h and direction N10°E.

b) Vector $0.8\vec{v}$ has magnitude $0.8 \times 30 = 24$ km/h and direction N10°E.

c) Vector $-\vec{v}$ has magnitude 30 km/h and direction S10°W (the exact opposite direction to N10°E).

d) Vector $-\frac{1}{2}\vec{v}$ has magnitude 15 km/h and direction S10°W.

Scalar Multiplication of Cartesian Vectors

If $\vec{v}$ is a vector described using coordinates in two-space or three-space, and k is a scalar, then the coordinates of $k\vec{v}$ can be calculated by multiplying each coordinate of $\vec{v}$ by k.

Examples

If $\vec{v}$ is the vector $(4, -2)$, calculate the coordinates and sketch each of the following vectors:

a) $1.5\vec{v}$

b) $-\vec{v}$

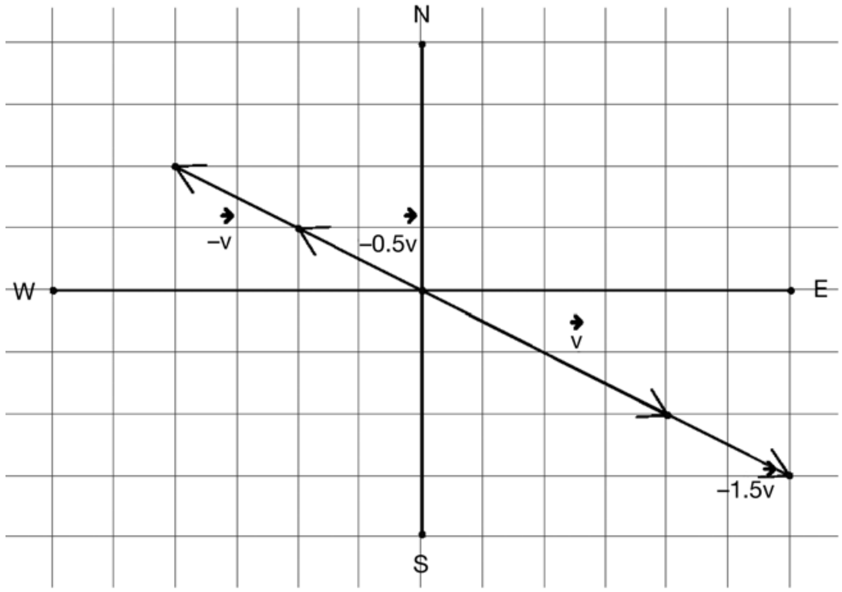
c) $-\frac{1}{2}\vec{v}$

Solutions

a) The vector $1.5\vec{v}$ is $1.5 \times (4, -2) = [1.5 \times 4, 1.5 \times (-2)] = (6, -3)$.

b) The vector $-\vec{v}$ is $(-1) \times (4, -2) = [(-1) \times 4, (-1) \times (-2)] = (-4, 2)$.

c) The vector $-\frac{1}{2}\vec{v}$ is $(-\frac{1}{2}) \times (4, -2) = [(-\frac{1}{2}) \times 4, (-\frac{1}{2}) \times (-2)] = (-2, 1)$.

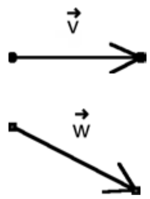


The following table is a summary of how to multiply Cartesian vectors in two-space and three-space:

Scalar multiplication of Cartesian vectors in two-space $\vec{v} = (a, b)$ $k\vec{v} = (ka, kb)$
Scalar multiplication of Cartesian vectors in three-space $\vec{v} = (a, b, c)$ $k\vec{v} = (ka, kb, kc)$

Support Questions
(do not send in for evaluation)

20. Given the following diagrams, sketch
    a) $3\vec{v}$
    b) $-2\vec{w}$



21. Calculate the coordinates for each of the following:
    a) (1, 2) + 3(–2, 1)
    b) (1, –2) – 2(3, –1)
    c) (1, 2, –3) + 2(1, –1, 2)

There are Suggested Answers to Support Questions at the end of this unit.

Properties of Vector Arithmetic

In Lesson 14, you learned about the commutative property of vector addition. Expressed in words, this property says that when two vectors are added, the order is unimportant. Expressed in symbols, it says that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for any two vectors $\vec{u}$ and $\vec{v}$.

Other properties or rules for adding and multiplying vectors have also been given special names. Conveniently, many of these rules are similar to the rules of algebra for dealing with real numbers that you should already be familiar with. (There are some exceptions. In a later lesson, you'll learn about something called the cross product of two vectors, where the order in which the vectors are listed makes a difference.) In this lesson, you'll focus on associative and distributive properties of vector arithmetic.

The Associative Property

Suppose three different forces are acting on an object, each with its own magnitude and direction. These three forces can be represented by three vectors: call them $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The overall or net effect of these three forces, also called the resultant vector, is the sum of these three vectors: $\vec{a} + \vec{b} + \vec{c}$.

The associative property says that this sum can be calculated in either of two ways:

- first find $\vec{a} + \vec{b}$, then add it to $\vec{c}$
- first find $\vec{b} + \vec{c}$, then add $\vec{a}$ to it

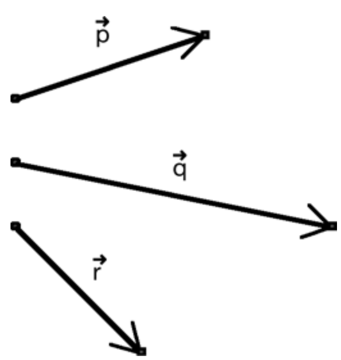
This property can be expressed in symbols:
$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

Example

Given the vectors in the diagram, find $\vec{p} + \vec{q} + \vec{r}$ in two ways:

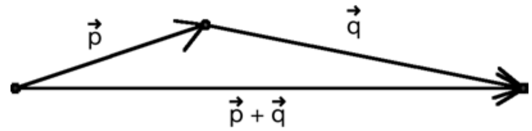
a) find $\vec{p} + \vec{q}$, then add it to $\vec{r}$

b) find $\vec{q} + \vec{r}$, then add $\vec{p}$ to it

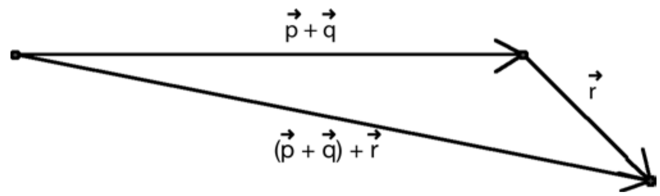


Solution

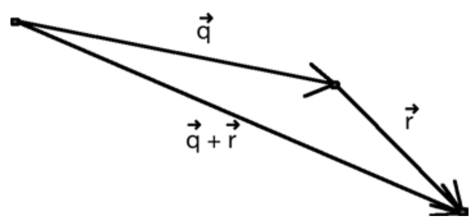
a) Find $\vec{p} + \vec{q}$ by putting the two vectors $\vec{p}$ and $\vec{q}$ together, tip to tail:



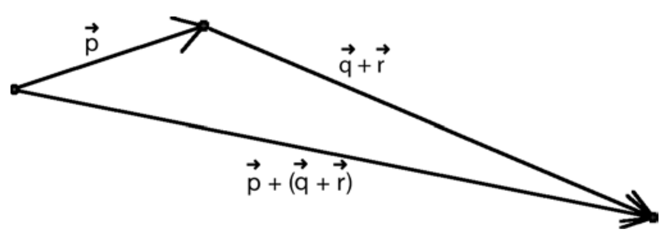
Then find $(\vec{p} + \vec{q}) + \vec{r}$ by putting the two vectors $(\vec{p} + \vec{q})$ and $\vec{r}$ together, tip to tail:



b) Find $\vec{q} + \vec{r}$ by putting the two vectors $\vec{q}$ and $\vec{r}$ together, tip to tail:



Then find $\vec{p} + (\vec{q} + \vec{r})$ by putting the two vectors $\vec{p}$ and $(\vec{q} + \vec{r})$ together, tip to tail:



Notice that if you compare the answers for a) and b), $(\vec{p} + \vec{q}) + \vec{r}$ and $\vec{p} + (\vec{q} + \vec{r})$ turn out to be equal vectors.

The Distributive Property

The distributive property is another rule that strongly resembles the rules of algebra for real numbers. This rule says that if c and d are any scalars and if $\vec{u}$ and $\vec{v}$ are any vectors, then $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ and $(c + d)\vec{u} = c\vec{u} + d\vec{u}$.

How can this property be used when calculating coordinates for specific vectors and scalars? Here are some examples.

Examples

Given two vectors $\vec{u} = (3, 2, -1)$ and $\vec{v} = (-1, 0, 1)$:

a) Find $c(\vec{u} + \vec{v})$ and $c\vec{u} + c\vec{v}$. Verify that these two vectors are equal.

b) Find $(c + d)\vec{u}$ and $c\vec{u} + d\vec{u}$. Verify that these two vectors are equal.

Solutions

a)
$$\begin{aligned} c(\vec{u} + \vec{v}) &= c[(3, 2, -1) + (-1, 0, 1)] \\ &= c(2, 2, 0) \\ &= (2c, 2c, 0) \end{aligned}$$

$$\begin{aligned} c\vec{u} + c\vec{v} &= c(3, 2, -1) + c(-1, 0, 1) \\ &= (3c, 2c, -c) + (-c, 0, c) \\ &= (2c, 2c, 0) \end{aligned}$$

Therefore, $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.

b)
$$\begin{aligned} (c + d)\vec{u} &= (c + d)(3, 2, -1) \\ &= (3(c + d), 2(c + d), -(c + d)) \end{aligned}$$

$$\begin{aligned} c\vec{u} + d\vec{u} &= c(3, 2, -1) + d(3, 2, -1) \\ &= (3c, 2c, -c) + (3d, 2d, -d) \\ &= (3(c + d), 2(c + d), -(c + d)) \end{aligned}$$

Therefore, $(c + d)\vec{u} = c\vec{u} + d\vec{u}$.

It has already been shown that vector addition and scalar multiplication satisfy the distributive property. You can use this to simplify expressions involving vectors. Here are some examples.

Examples

Simplify the following expressions:

a) $\vec{x} - 2(\vec{x} + 3\vec{y})$

b) $-4\vec{x} + 2(3\vec{x} - 2\vec{y} + 4\vec{x}) + 6\vec{x}$

c) $6(3\vec{x} - 2\vec{y}) + 4(3\vec{x} - 5\vec{y}) - 5(2\vec{x} - 5\vec{y})$

Solutions

Using the distributive law, these expressions can be simplified:

a) $\vec{x} - 2(\vec{x} + 3\vec{y})$

$= \vec{x} - 2\vec{x} - 6\vec{y}$

$= -\vec{x} - 6\vec{y}$

b) $-4\vec{x} + 2(3\vec{x} - 2\vec{y} + 4\vec{x}) + 6\vec{x}$

$= -4\vec{x} + 6\vec{x} - 4\vec{y} + 8\vec{x} + 6\vec{x}$

$= 16\vec{x} - 4\vec{y}$

c) $6(3\vec{x} - 2\vec{y}) + 4(3\vec{x} - 5\vec{y}) - 5(2\vec{x} - 5\vec{y})$

$= 18\vec{x} - 12\vec{y} + 12\vec{x} - 20\vec{y} - 10\vec{x} + 25\vec{y}$

$= 20\vec{x} - 7\vec{y}$

The following table summarizes the associative and distributive properties for vectors:

Suppose $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors, and c and d are any scalars.
Associative property $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
Distributive property $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ and $(c + d)\vec{u} = c\vec{u} + d\vec{u}$

Support Question
(do not send in for evaluation)

22. Simplify the following expressions:
    a) $\vec{x} - 3(\vec{x} + 4\vec{y})$
    b) $-3\vec{x} + 2(5\vec{x} - \vec{y} + 3\vec{x}) + 7\vec{x}$
    c) $5(2\vec{x} - 3\vec{y}) + 4(3\vec{x} - 2\vec{y}) - 6(2\vec{x} - 3\vec{y})$

Solving Real-World Problems Using Vectors

There are many physical real-world problems that can be modelled using addition or subtraction of vectors. Most typically, these problems involve several different forces or velocities acting on an object, and you are interested in the overall or net effect.

Example

An airplane heads in the direction W35°N at a speed of 800 km/h. The wind is blowing at 65 km/h in the direction N25°E. Determine the resultant velocity of the plane.

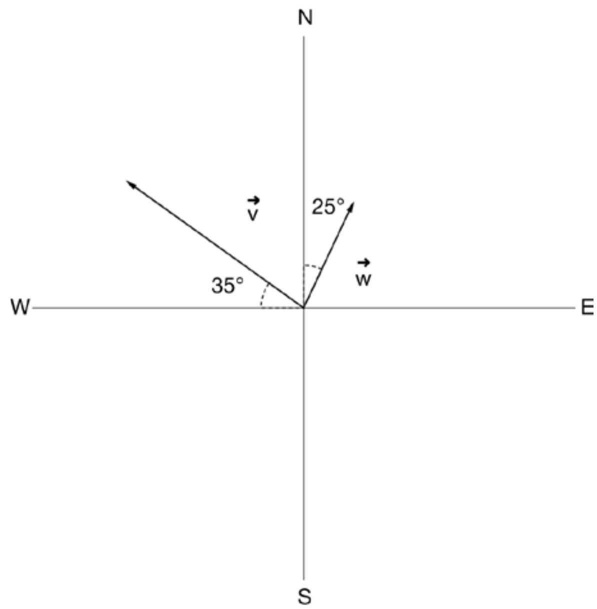
Solution

Let $\vec{v}$ be the vector with direction W35°N and magnitude 800 km/h, which represents the velocity that the airplane is steering.

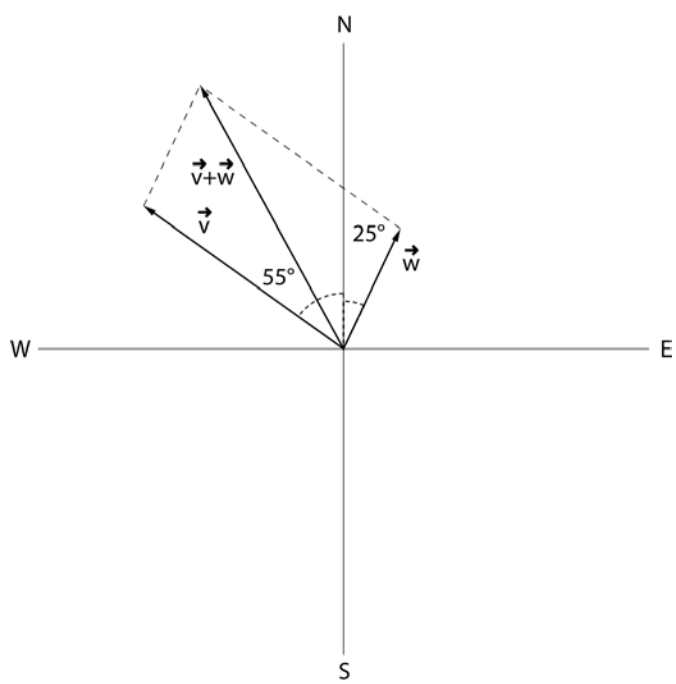
Let $\vec{w}$ be the vector with direction N25°E and magnitude 65 km/h, which represents the velocity of the wind.

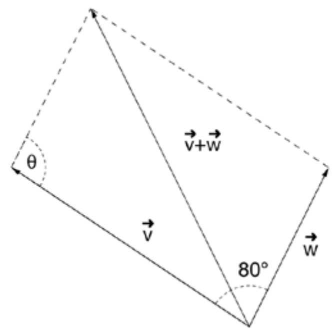
The resultant velocity of the plane is the net effect of these two velocities on the plane, which can be calculated by adding the vectors $\vec{v}$ and $\vec{w}$ together.

First, draw the vectors so that they can be added together.



Complete the parallelogram and use the procedure you used in Lesson 14 to find the sum of the two vectors.





$\theta = 180 - 80 = 100°$

Next, find the magnitude of $\vec{v} + \vec{w}$ using the cosine law:

$|\vec{v} + \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta$

$|\vec{v} + \vec{w}|^2 = 800^2 + 65^2 - 2 \times 800 \times 65 \times \cos 100°$

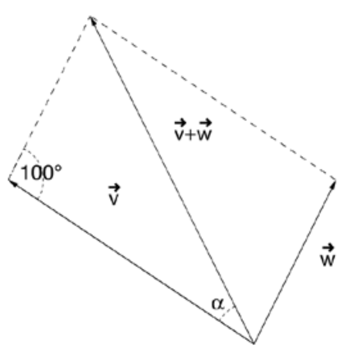
$|\vec{v} + \vec{w}|^2 = 640\,000 + 4225 - 104\,000 \times (-0.1736)$

$|\vec{v} + \vec{w}|^2 = 662\,279.4$

$|\vec{v} + \vec{w}| = \sqrt{662\,279.4}$

$|\vec{v} + \vec{w}| = 813.8$

Finally, find the direction of $\vec{v} + \vec{w}$ by finding the angle separating it from $\vec{v}$ using the sine law:



$$\frac{\sin\alpha}{|\vec{w}|} = \frac{\sin 100°}{|\vec{v} + \vec{w}|}$$

$$\frac{\sin\alpha}{65} = \frac{0.9848}{813.8}$$

$$\sin\alpha = \frac{0.9848}{813.8} \times 65$$

$$\alpha = \sin^{-1}(0.0787)$$

$$\alpha = 4.5°$$

Add this value to the direction of $\vec{v}$ to calculate the direction of $\vec{v} + \vec{w}$. The resultant velocity of the airplane is 813.8 km/h in the direction W39.5°N.

You might also encounter a similar problem where the resultant velocity is specified.

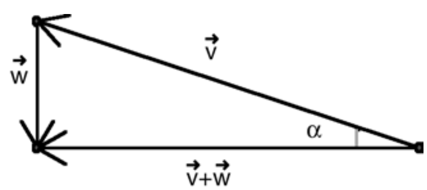
Example

A pilot wishes to fly to a city 60 km west. She finds that she must steer the plane slightly north of due west, because wind is blowing from the north at 50 km/h. In what direction must she steer the plane if her speedometer reads 600 km/h?

Solution

Let $\vec{v}$ represent the velocity at which she flies the plane, and let $\vec{w}$ represent the velocity of the wind.

You know $\vec{w}$ has a magnitude of 50 km/h and a direction of due south, and you know $\vec{v}$ has a magnitude of 600 km/h. You need to find the direction of $\vec{v}$.



The vectors $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$ form a right-angled triangle. You need to find the angle α. Since you know the length of the side opposite α and the length of the hypotenuse, you can find α using the definition of the sine function:

$$\sin\alpha = \frac{opposite}{hypotenuse}$$

$$\sin\alpha = \frac{50}{600}$$

$$\alpha = \sin^{-1}(0.0833)$$

$$\alpha = 4.78°$$

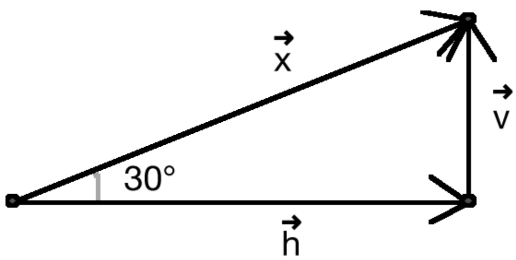
The pilot should steer in the direction W4.78°N.

For some problems, it is useful to separate a force into horizontal and vertical components. In other words, given a vector, you can express it as a sum of two vectors where one is horizontal and one is vertical.

Example

A force of 200 N acts in a direction forming an angle of 30° with the horizontal. Find the magnitudes of the horizontal and vertical components of the force.

Solution



Let $\vec{x}$ represent the 200 N force forming an angle of 30° with the horizontal, and let $\vec{h}$ and $\vec{v}$ represent its horizontal and vertical components respectively. You need to find the magnitude of $\vec{h}$ and $\vec{v}$.

The three vectors form a right-angled triangle, and you know the angle between $\vec{x}$ and $\vec{h}$. You also know the length of the hypotenuse is 200. Thus, you can find the lengths of the other vectors using the definitions of the sine and cosine functions.

The length of $\vec{v}$ is calculated as follows:

$$\sin 30° = \frac{|\vec{v}|}{|\vec{x}|} = \frac{|\vec{v}|}{200}$$

$$200\sin 30° = |\vec{v}|$$

$$|\vec{v}| = 100$$

The length of $\vec{h}$ is calculated as follows:

$$\cos 30° = \frac{|\vec{h}|}{|\vec{x}|} = \frac{|\vec{h}|}{200}$$

$$200\cos 30° = |\vec{h}|$$

$$|\vec{h}| = 173.2$$

The horizontal and vertical components have magnitudes of 173.2 N and 100 N respectively.

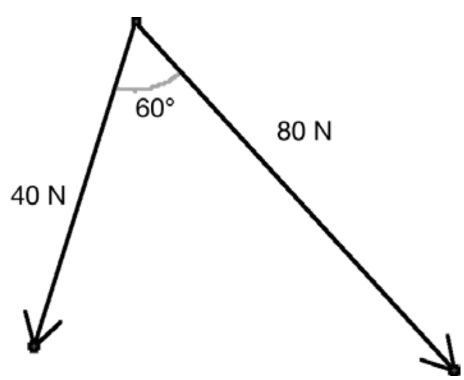
Another application of vectors is to calculate forces acting on an object. If a number of forces are acting on an object, you can replace them with a resultant force that has the same effect. Here are some examples.

Example

Two forces of magnitudes 40 N and 80 N act on an object. The angle between the forces is 60°. Find the magnitude of the resultant force.

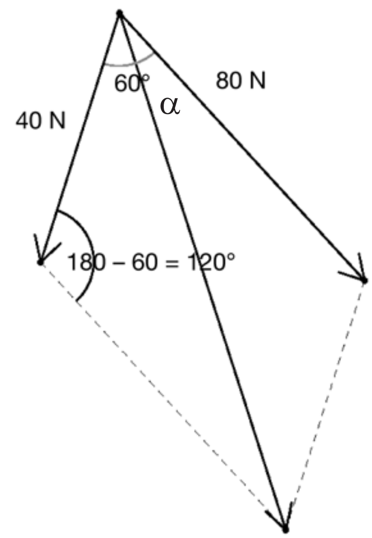
Solution

Draw two vectors representing the forces with an angle of 60° separating them.



Use the parallelogram method to find the resultant of the two forces.

Complete the parallelogram:



Let $\vec{r}$ be the resultant of the two forces.

$$|\vec{r}|^2 = 40^2 + 80^2 - 2(40)(80)\cos 120$$
$$= 1600 + 6400 - 6400(-0.5)$$
$$= 11\,200$$

$$|\vec{r}| = 105.83$$

$$\frac{\sin\alpha}{40} = \frac{\sin 120}{105.83}$$
$$\sin\alpha = \frac{40\sin 120}{105.83}$$
$$\sin\alpha \approx .33$$
$$\alpha = 19.27°$$

The resultant of the two forces is 105.83 N at an angle of 19.27° from the 80 N force.

Support Questions
(do not send in for evaluation)

23. A boat heads east with a speed of 20 km/h, but there is a current with a speed of 5 km/h in the direction S20°E. Find the resultant velocity of the boat.

24. Two forces are acting on an object. The first force has a magnitude of 6 N and is acting toward the southwest. The second force has a magnitude of 8 N and is acting toward the northwest. Find the resultant force.

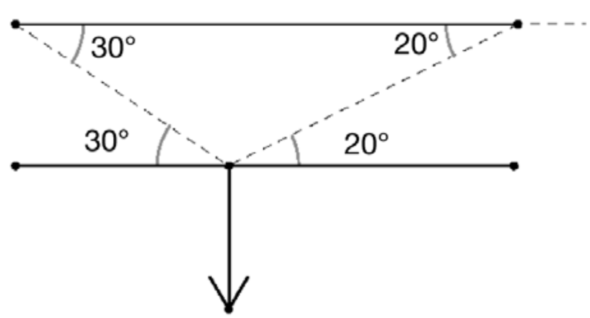
The final example of this lesson is an application of vectors to the calculation of tension.

Example

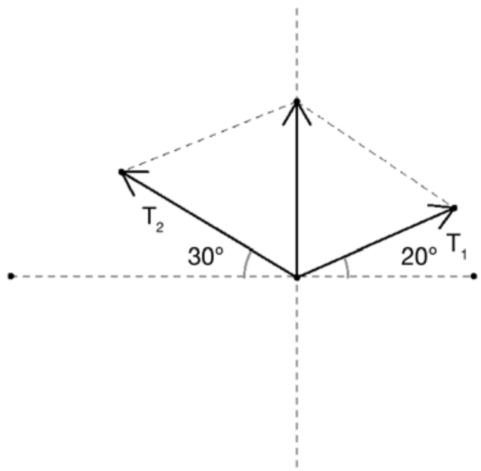
A 100 N chandelier is suspended from a ceiling at a single point by two chains that make angles of 20° and 30° with the ceiling. Calculate the tension on each chain.

Solution

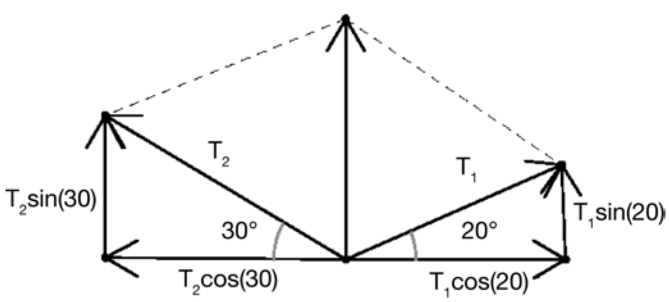
Start with a diagram of the chains:



Draw a diagram of the forces acting on the chandelier. T_1 and T_2 represent the tension (the force) on the chains that keep the chandelier from falling.



The resultant of T_1 and T_2 must be equal and opposite to the weight of the chandelier. The horizontal components of T_1 and T_2 must be opposite and equal. The vertical component must equal 100 N.



Horizontal components:

$$\begin{aligned} T_1\cos(20) &= T_2\cos(30) \\ T_1 &= \frac{T_2\cos(30)}{\cos(20)} \\ T_1 &= 0.92T_2 \end{aligned}$$

Vertical components:

$$T_1\sin(20) + T_2\sin(30) = 100$$

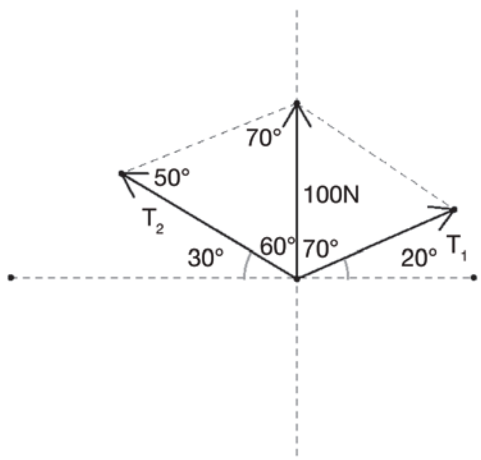
Substitute $T_1 = 0.92T_2$ into $T_1 \sin(20) + T_2 \sin(30) = 100$.

$0.92T_2 \sin(20) + T_2 \sin(30) = 100$
$0.31T_2 + 0.5T_2 = 100$
$0.81T_2 = 100$
$T_2 \approx 123.46 \text{ N}$

$T_1 = 0.92T_2$
$T_1 = 0.92(123.46)$
$T_1 \approx 113.58 \text{ N}$

The tensions in the chains are approximately 123.46 N and 113.58 N.

Alternate Solution to Example on page 17:



The resultant of T_1 and T_2 is equal and opposite to the weight of the chandelier.

Using the sine law:

$$\frac{100}{\sin 50} = \frac{T_2}{\sin 70} = \frac{T_1}{\sin 60}$$

$$T_2 = \frac{100 \sin 70}{\sin 50} \approx 122.67 \text{ N} \quad \text{and} \quad T_1 = \frac{100 \sin 60}{\sin 50} \approx 113.05 \text{ N}$$

The difference in the answers is due to rounding off values to two decimal places in the given solution.

Support Question
(do not send in for evaluation)

25. A 200 N object is suspended at a single point by two ropes that make angles of 35° and 45° with the ceiling. Calculate the tension on each rope.

Conclusion

In this lesson, you learned some properties of vectors, and how to multiply a vector by a scalar. In Lesson 16, the first one in Unit 4, you will explore other ways in which two vectors can be multiplied by each other.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(27 marks)

41. Simplify the expressions: **(2 marks: 1 mark each)**

    a) $3(2\vec{u} - 3\vec{v}) - 2(-2\vec{v} + 3\vec{u})$

    b) $5(\vec{a} - 2\vec{b} + 4\vec{c}) - 4(2\vec{a} - 3\vec{b} - 5\vec{c})$

42. A plane is flying E40°N at a speed of 600 km/h. The wind is blowing at 10 km/h in the direction N20°W. Determine the resultant velocity of the plane. **(6 marks)**

43. Two forces of magnitude 100 N and 70 N act on an object. The angle between the forces is 120°. What is the resultant force? **(6 marks)**

44. Find the horizontal and vertical components of a force of 75 N that acts in a direction forming an angle of 51° with the vertical. **(5 marks)**

45. A weight of 125 N is suspended from a ceiling at a single point by two cords that make angles of 30° and 45° with the ceiling. Calculate the tension in each cord. **(8 marks)**

This is the last lesson in Unit 3. When you have finished, do the Reflection for Unit 3. Follow any other instructions you have received from ILC about submitting your coursework, then send it to ILC. A teacher will mark your work and you will receive your results online.