MCV4U-A

13

Introduction to Vectors

Introduction

So far in this course, you have focused on calculus, specifically calculating rates of change and finding derivatives. You now turn your attention to vectors.

Did you know that vectors can be applied to computer-game design, economics, genetics, sociology, and cryptography? In this lesson, you will explore real-life situations that can be represented by vectors. You will use vectors to represent quantities and look at various ways to represent vectors.

Estimated Hours for Completing This Lesson	
Introduction to Vectors	1
Equal Vectors	0.5
Representation of Vectors	2
Key Questions	1.5

What You Will Learn

After completing this lesson, you will be able to

- recognize a vector as a quantity with both a magnitude and a direction
- identify, gather, and interpret information about real-world applications of vectors
- represent a vector in two-space geometrically as a directed line segment, with directions expressed in different ways
- represent a vector algebraically using Cartesian and polar coordinates

Introduction to Vectors

Many of the problems you have solved in mathematics involve quantities, such as the cost of a product, the area of a square, and the volume of a can. These are scalars, quantities that can be represented by a specific number. The value of a scalar is also called its magnitude.

Certain quantities need more than just a single value to represent them. These are referred to as vectors. Examples of vectors include forces, velocities, and acceleration. Consider the following examples:

- a plane flying east at 400 km/h
- a force of 40 N pulling a bicycle forward

Vector quantities require more than a single scalar (number) to represent them. They are defined as follows:

A vector is a quantity that has both a magnitude and a direction.

Are the following examples of scalars or vectors?

- the current time
- the size of a computer hard drive
- the speed of a train

These are all scalars because they are represented by a value that has no direction attached to it.

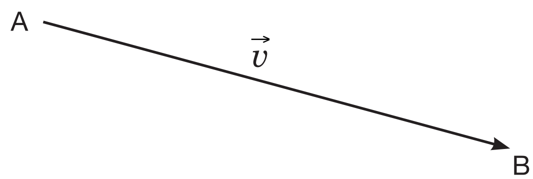
What about a person's weight? Your first guess may be that it's a scalar because you think of it in terms of a value, but it is in fact a vector. A person's weight is the force exerted as a result of gravity and so has both a magnitude and a direction.

Try another example. Do you think you can represent wind with a vector? Yes, wind has a direction and a magnitude. For example, the wind is blowing from the east at a speed of 30 km/h.

Vectors are represented geometrically and algebraically. This helps you develop tools to solve problems involving vectors.

Geometric Vectors

A vector is represented geometrically as an arrow (a directed line segment).



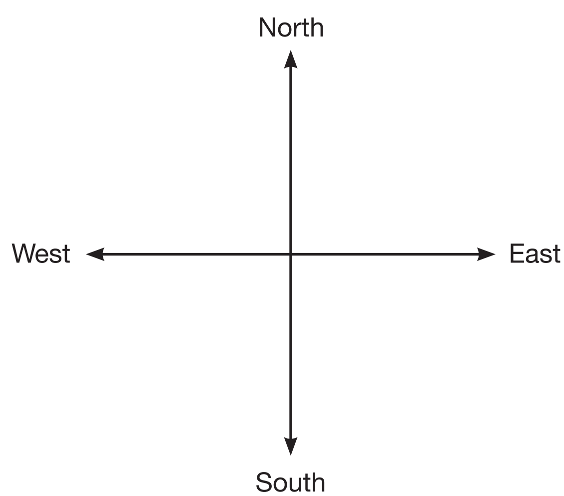
To describe this vector, write it as $\vec{AB}$. The tail of the vector is A and the tip is B. You can also label it $\vec{v}$. Any vector that has the same direction and magnitude as $\vec{AB}$ is equal to $\vec{AB}$. The magnitude of $\vec{AB}$ is the length of the line segment $\vec{AB}$. It is written as $|\vec{AB}|$ or $|\vec{v}|$, and it is always a positive number.

Look at an example:

An airplane flying east at 400 km/h can be represented by a directed line segment (arrow) with the length equivalent to 400 units. Is this enough to represent the vector of the plane? No, you still need to represent the direction of the plane.

Describing the Direction of a Vector

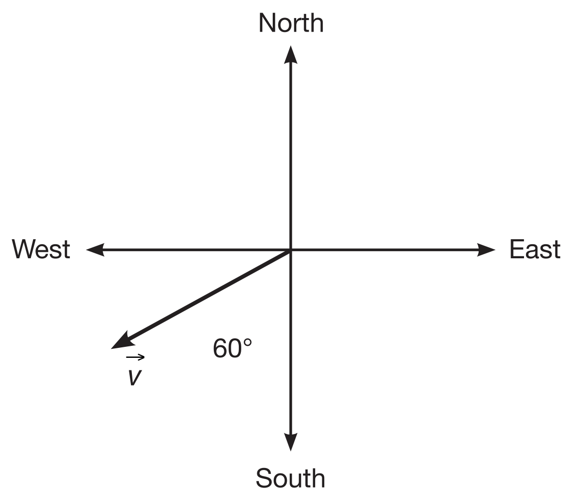
You can use the compass method to represent the direction of a vector. In this method, the direction is given by measuring the angle clockwise from a reference direction, usually north.



A plane heading south, for example, has a 180° bearing.

Continuing the example, if the airplane is travelling east, the vector is represented by 90° using the compass method.

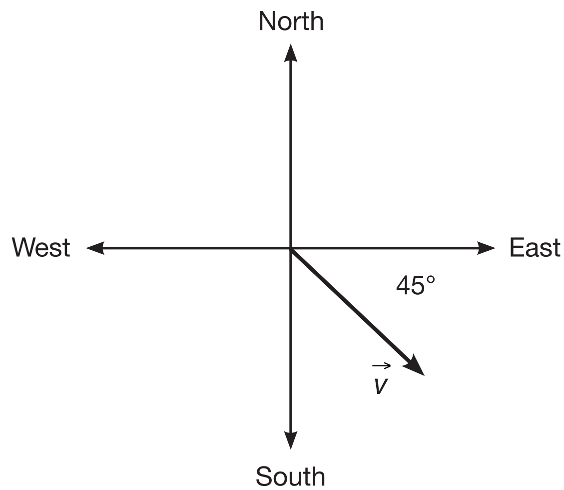
Now look at another example. The vector $\vec{v}$ has a bearing of 240° (180° + 60°). Remember the direction of the vector is measured starting from north and in a clockwise direction.



The four principal directional indicators—north, east, south, and west—are called cardinal directions. Another way to describe the direction is to measure the angle starting at one of the cardinal directions toward another. In the example, the direction of the vector $\vec{v}$ can be written as W30°S, which means start at west and rotate 30° toward south. You can also write it as S60°W since starting at south and rotating 60º toward west gives the same result.

Example

For the following vector, take a moment to think about four different ways to represent its direction:



Solution

The direction of the vector can be represented in the following ways:

- 135° (using the compass method, measure clockwise from north to get a bearing of 90° + 45° = 135°)
- E45°S (starting from the cardinal direction east, rotate 45° toward south)
- S45°E (starting from the cardinal direction south, rotate 45° toward east)
- W135°E (starting from the cardinal direction west, rotate 135° toward east)

Support Questions
(do not send in for evaluation)

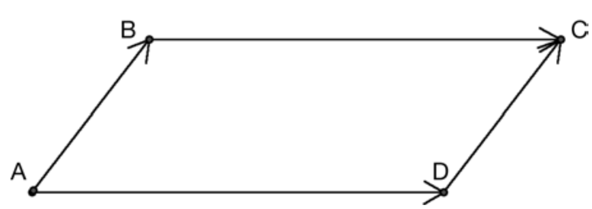
7. Can the following quantities be represented by a vector? Justify your answer.
   a) Height
   b) Current in a river
   c) Velocity of an airplane

8. Draw vectors to represent the following and express its direction using an alternative notation:
   a) A displacement of 60 km southeast
   b) A weight of 35 N acting vertically downward
   c) A velocity of 150 m/s on a bearing of 188°

There are Suggested Answers to Support Questions at the end of this unit.

Equal Vectors

Two vectors are equal if they have the same direction and equal magnitudes. In the following parallelogram, the vectors $\vec{AB}$ and $\vec{DC}$ are two equal vectors.



Opposite vectors have equal magnitude and are parallel, but they have opposite orientation. The vectors $\vec{AB}$ and $\vec{CD}$ on the parallelogram are opposite vectors.

Using the parallelogram, state another pair of equal vectors and another pair of opposite vectors.

$\overrightarrow{BC} = \overrightarrow{AD}$

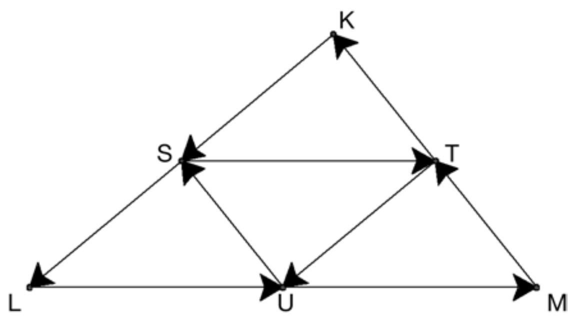
$\overrightarrow{BC}$ and $\overrightarrow{DA}$ are opposites.

Support Question

(do not send in for evaluation)

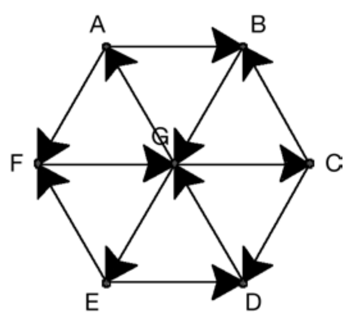
9. Use the geometric properties of vectors to list all pairs of equal vectors for the following:

a)



S, *T*, and *U* are midpoints of segments *KL*, *KM*, and *LM*.

b)



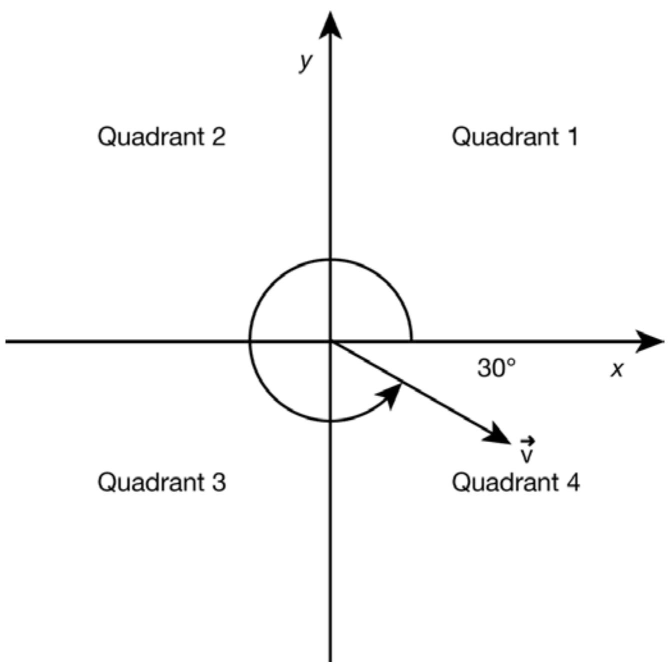
ABCDEF is a regular hexagon.

Representations of Vectors

You can represent vectors algebraically using either polar form or Cartesian form.

Polar Form

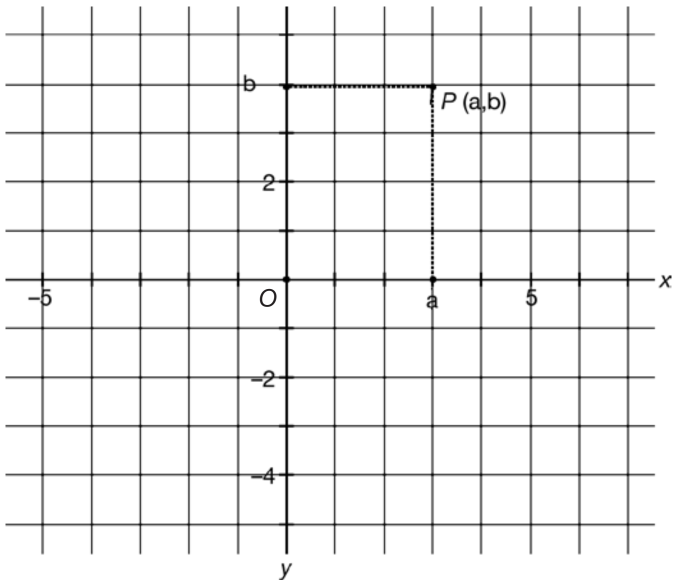
You already know that a vector is a directed line segment with both a magnitude and a direction. You can use $|\vec{v}|$ to represent the magnitude of a vector $\vec{v}$. To describe the direction, use the angle measured counter-clockwise from the positive side of the x-axis. For example:



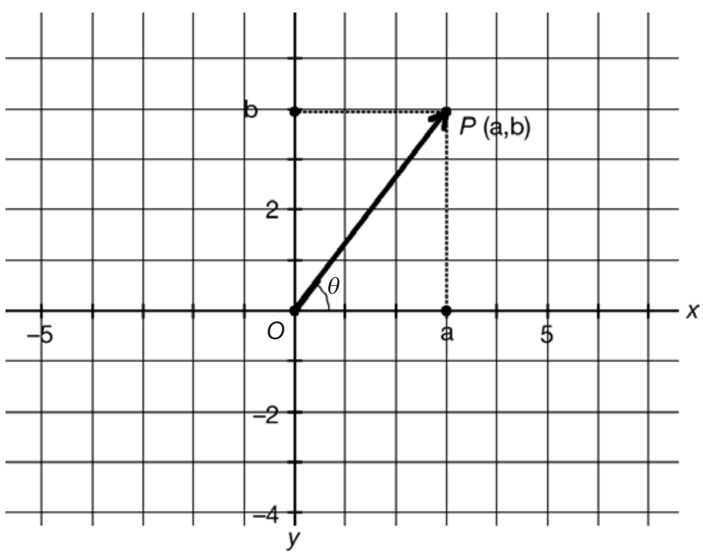
The direction of $\vec{v}$ is $360° - 30° = 330°$.

Cartesian Representation

A two-dimensional Cartesian system can be formed when you draw two perpendicular axes that intersect at an origin, which is labelled O. Each point in the plane has two coordinates, referred to as an ordered pair and written (a, b), where a is the x-coordinate and b is the y-coordinate.



Given any vector $\vec{u}$ in the plane, you can move $\vec{u}$ until its tail is at the origin (O), the tip of the vector $\vec{u}$ will be at some P with coordinates (a, b). You would write $\vec{u} = (a, b)$, a is the x-component of $\vec{u}$ and b is the y-component of $\vec{u}$. To find the magnitude and direction of $\vec{u}$, you need to do the following:



Step 1: Magnitude

The Pythagorean theorem tells you that

$$|OP|^2 = a^2 + b^2$$

$$|OP| = \sqrt{a^2 + b^2}$$

The magnitude of $\vec{u}$ is $\sqrt{a^2 + b^2}$.

Step 2: Direction

To find the angle, use the tangent ratio:

$$\begin{aligned}\tan\theta &= \frac{opposite}{adjacent} \\ &= \frac{b}{a}\end{aligned}$$

Use your calculator to find θ, whose tangent is $\frac{b}{a}$. To do this, you usually need to use a combination of 2nd function and tan buttons on your calculator.

Examples

Find the magnitude and direction of each of the following vectors:

a) (2, –1)

b) (–1, 3)

Solutions

a)

Step 1: Magnitude

Let $\vec{u} = (2, -1)$.

$|\vec{u}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$

The magnitude of $\vec{u} = \sqrt{5}$.

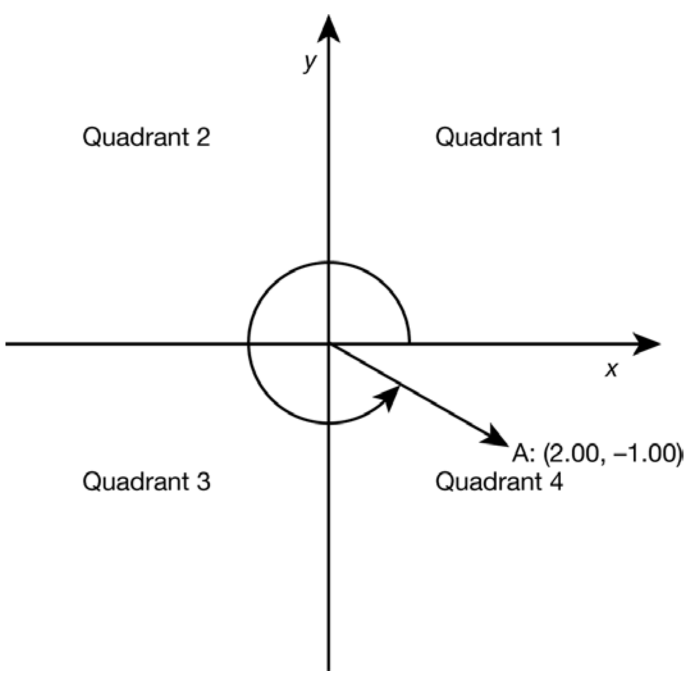
Step 2: Direction

$$\tan\theta = \frac{-1}{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{2}\right)$$

Using a scientific calculator, you get $\theta \cong -26.6°$.

Since the terminal point (2, –1) is in the fourth quadrant, you can conclude that the vector makes an angle of 360° – 26.6° = 333.4° with the positive *x*-axis, measured counter-clockwise.



b)

Step 1: Magnitude

Let $\vec{v} = (-1, 3)$

$|\vec{v}| = \sqrt{(-1)^2 + (3)^2} = \sqrt{10}$

Step 2: Direction

$\theta = \tan^{-1}\left(\dfrac{3}{-1}\right)$

Using a scientific calculator, you get $\theta \cong -71.5°$.

Since the terminal point of the vector $(-1, 3)$ is in the second quadrant, you can conclude that the vector makes an angle of $180° - 71.5° = 108.5°$ with the positive x-axis, measured counter-clockwise.

In the previous examples, you solved for the magnitude and direction of a vector given its Cartesian components. In the next examples, you will learn to do the reverse. You are given the magnitude and direction and asked to find the Cartesian components.

Examples

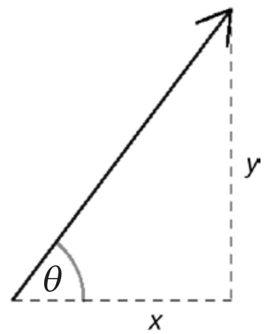
Write each of the following vectors in Cartesian form:

a) $|\vec{u}| = 6$ and $\theta = 30°$

b) $|\vec{v}| = 12$ and $\theta = 90°$

c) $|\vec{w}| = 20$ and $\theta = 120°$

Solutions



The diagram illustrates the x- and y-components of a vector.

Use a standard scientific calculator to do the calculation.

a)

The x-component:

$$\begin{aligned} x &= 6\cos(30) \\ &= 5.20 \end{aligned}$$

The y-component:

$$\begin{aligned} y &= 6\sin(30) \\ &= 3 \end{aligned}$$

The vector in Cartesian form is $\vec{u} = (5.20, 3)$.

b)

The x-component:

$$\begin{aligned} x &= 12\cos(90) \\ &= 0 \end{aligned}$$

The y-component:

$$\begin{aligned} y &= 12\sin(90) \\ &= 12 \end{aligned}$$

The vector in Cartesian form is $\vec{v} = (0, 12)$.

c)

The x-component:

$x = 20\cos(120)$

$= -10$

The y-component:

$y = 20\sin(120)$

$= 17.3$

The vector in Cartesian form is $\vec{w} = (-10, 17.3)$.

Support Questions

(do not send in for evaluation)

10. Find the magnitude and direction of each of the following vectors:

    a) $(1, -2)$

    b) $(-1, 6)$

    c) $\left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$

11. Write each of the following vectors in Cartesian form:

    a) $|\vec{u}| = 10$ and $\theta = 30°$

    b) $|\vec{v}| = 12$ and $\theta = 240°$

    c) $|\vec{w}| = 8$ and $\theta = 330°$

    d) $|\vec{x}| = 16$ and $\theta = 135°$

Conclusion

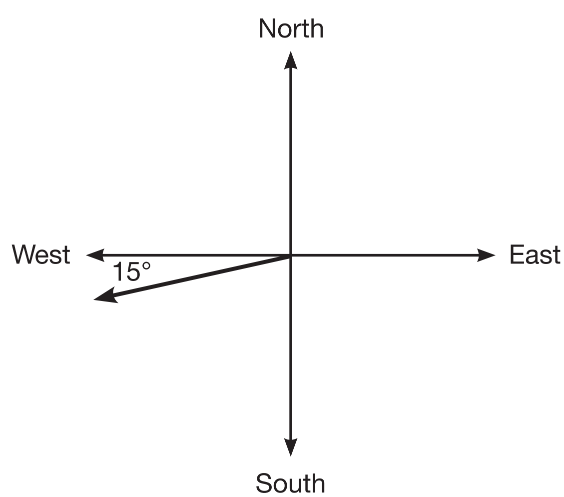
In this lesson, you learned about vectors and how to represent them. In Lesson 14, you will be introduced to vectors in a three-dimensional space and learn how to add and apply other operations to vectors.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(16 marks)

34. Describe the direction of the vector in four different ways: **(2 marks)**



35. Find the magnitude and direction of each of the following vectors: **(5 marks: 2½ marks each)**

a) (–8, 8)

b) (2, –6)

36. Write each of the following vectors in Cartesian form:
(9 marks: 3 marks each)

a) $|\vec{u}| = 6$ and $\theta = 45°$

b) $|\vec{v}| = 12$ and $\theta = 150°$

c) $|\vec{u}| = 4$ and $\theta = 90°$

Now go on to Lesson 14. Do not submit your coursework to ILC until you have completed Unit 3 (Lesson 11 to 15).

