MCV4U-A

Unit

3

Applying Derivatives and Introduction to Vectors

December 6, 2010

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Introduction

In this unit, you will complete the calculus portion of the course and begin looking at vectors.

In the first two lessons, you will learn about applications of derivatives. For example, suppose you want to find the minimum amount of material to use to build a box of a specific volume. This is called an optimization problem. You learned earlier in the course that derivatives help you study the rate of change and how the change in one variable leads to the change in a dependent variable. You also learned that the derivative of a function helps determine the maximum and minimum of a function over an interval. In this unit, you will learn how to use your knowledge of calculus to solve a number of optimization problems from the world of science and economics.

The last three lessons of this unit introduce you to the concept of a vector. You will develop the knowledge and skills necessary to solve problems involving velocities and forces represented as vectors.

Overall Expectations

After completing this unit, you will be able to

- solve problems from the world of economics, physics, and population growth using your knowledge of derivatives
- use your knowledge of derivatives to answer real-world optimization problems involving polynomial functions, rational functions, and exponential problems
- identify, gather, and interpret information about real-world applications of vectors
- represent a vector in two-space geometrically and algebraically
- perform vector addition and subtraction on vectors represented as directed line segments in two-space and vectors in Cartesian form in two-space and three-space
- solve real-world problems involving vector addition, vector subtraction, and scalar multiplication

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Real-Life Applications

Introduction

You are now familiar with the concept of the derivative as a rate of change, and you have spent much time learning how to compute the derivatives of various functions. In this lesson, you will see how your familiarity with derivatives can be applied to answer various real-life problems that involve rates of change.

Estimated Hours for Completing This Lesson	
Marginal Cost	1.5
Half-Life	1
Population Growth	1
Key Questions	1.5

For this lesson, there is an interactive tutorial on your course page. You may find it helpful during, or at the end of, this lesson to work through the tutorial called "Rate of Change Applications."

What You Will Learn

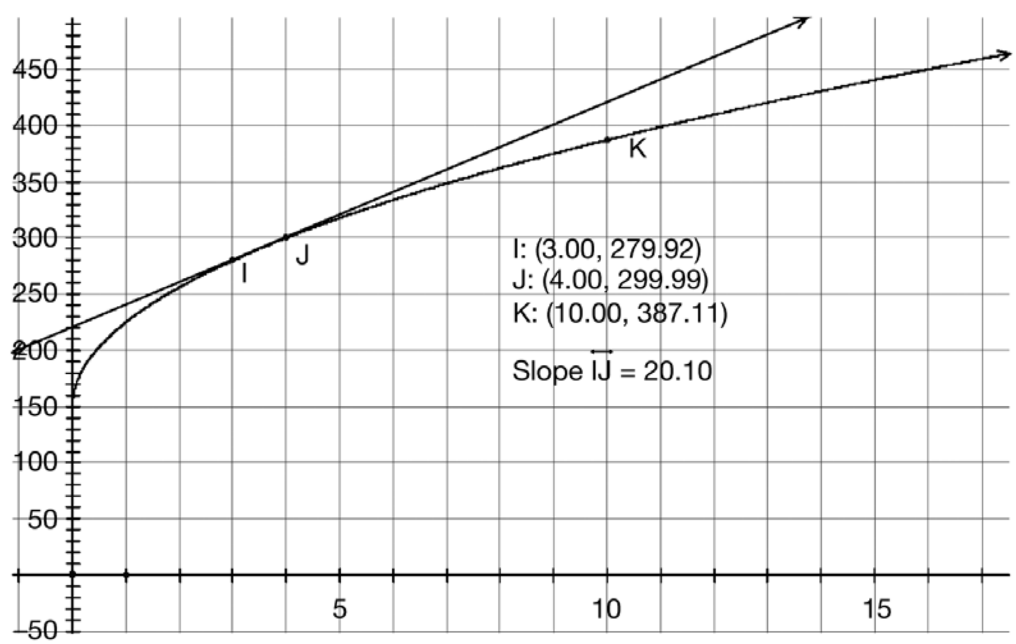
After completing this lesson, you will be able to

- interpret and answer questions from the world of economics (marginal cost), physics (half-life), and population growth using your knowledge of derivatives
- solve some problems involving related rates using your knowledge of derivatives

Marginal Cost

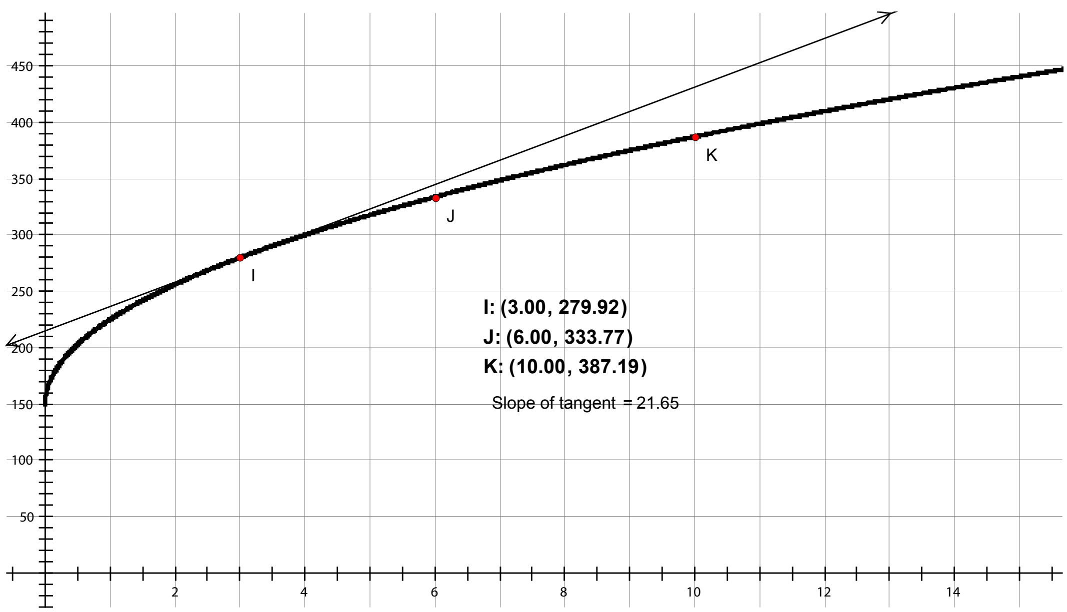
One of the many real-world applications where derivatives are relevant is in the fields of business and economics. Something commonly known as "marginal cost" is a derivative. Here is an example:

Imagine Pat owns a small business that manufactures tennis rackets. In the following graph, the variable y represents how much it costs in total to make tennis rackets as a function of x, where x is the number of rackets produced. If it costs Pat \$279.92 to make three rackets (I), it costs Pat \$299.99 to make four rackets (J) and \$387.11 to make ten rackets (K).



The variable y is measured in dollars and the variable x is measured in number of rackets. (Notice that on this graph the variable x can be any real number. For example, it's possible to have a point on the graph where x is 3.17. It might not seem to make sense to have 3.17 tennis rackets, but it is common when modelling real-world situations to allow some variables to be any real number.)

For the next graph, as for any other graph, it is meaningful to talk about the slope of the tangent line at various points. You know the slope of the tangent line is given by the derivative, which can also be interpreted as a rate of change.



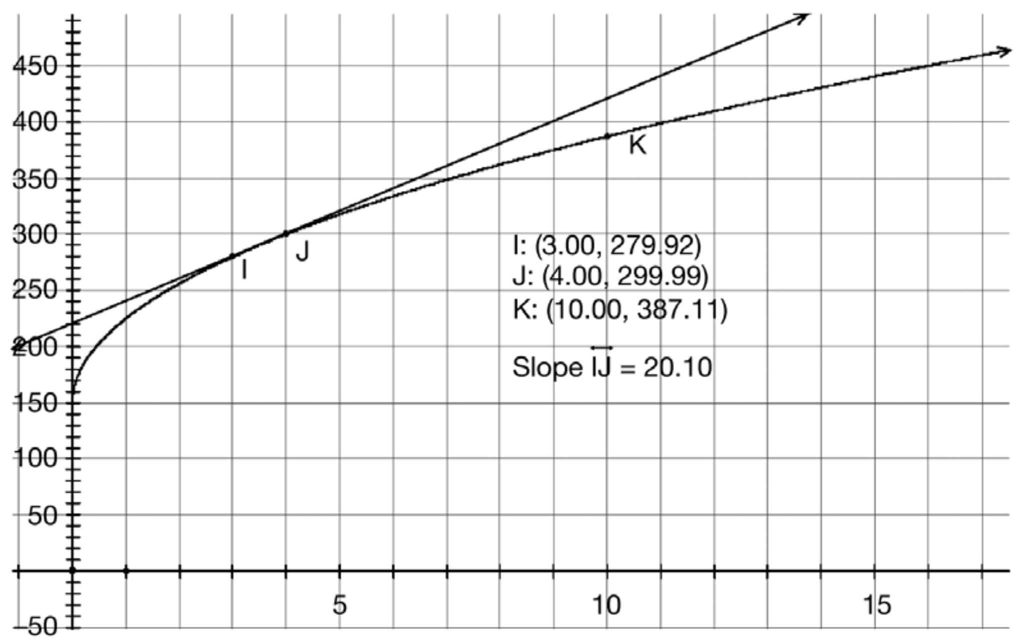
In this diagram, you see that the slope of the tangent is 21.65 when $x = 3$. In other words, $y' = 21.65$ when $x = 3$. If the units of the variable y are dollars and the units of the variable x are rackets, the derivative $y'(x)$ represents an instantaneous rate of change whose units are dollars per racket. It is interpreted as follows: At the moment in time when Pat is making 3 tennis rackets, the "instantaneous" cost per racket at that moment is 21.65 dollars per racket.

This doesn't mean that every racket costs $21.65 to produce (in fact, you know the cost to produce 3 rackets is $279.92). It means that, at that moment in time, Pat's costs are increasing at a rate of 21.65 dollars per racket.

Perhaps speaking of an instantaneous rate of change is not completely intuitive in this example. As you've seen in the past, an instantaneous rate of change of a function can be envisioned as the slope of the tangent line at that point. By contrast, you have also discussed the average rate of change over an interval, which can be envisioned as the slope of the secant line passing through two points.

For instance, on the graph, let I and J be the points where $x = 3$ and $x = 4$ respectively. The average rate of change of y over the interval (3, 4) is $\dfrac{299.99 - 279.92}{4 - 3} = 299.99 - 279.92 = 20.07$. (The graph generated by computer says 20.10, due to rounding.)

Since this is the difference between the cost of producing 4 rackets and the cost of producing 3 rackets, you could think of the slope of this secant line as representing the cost of producing the fourth racket:



Instead of the secant line, it is common to regard the slope of the tangent line at, say, $x = 3$ as the cost of producing the third racket, because derivatives are easier to compute with than slopes of secant lines. Moreover, for a smooth curve, the tangent line is an excellent approximation to the secant line joining two nearby points. In general, the derivative of the total-cost function at a given point is called the marginal cost. Since $y' = 21.65$ when $x = 3$, and $y' = 15.30$ when $x = 6$, you can informally say that at the moment when Pat is producing 3 rackets, the cost is \$21.65 per new racket. At the moment when Pat is producing

6 rackets, the cost is \$15.30 per new racket. You can say that the marginal cost of producing a tennis racket is \$21.65 when $x = 3$ and \$15.30 when $x = 6$.

Here's a summary:

> **Marginal Cost**
> If $f(x)$ is a function that represents the total cost of producing x items, then the derivative $f'(x)$ is called the marginal-cost function and can be regarded as the cost of producing the xth item.

Now consider an example where you have an equation to work with.

Example

Suppose Pat determines that the cost of manufacturing x tennis rackets is given by $f(x) = 150 + 75\sqrt{x}$.

a) What is the average cost of producing 100 tennis rackets?

b) What is the marginal cost of producing the 100th racket?

c) If Pat sells the tennis rackets for \$7.50 each, what function represents Pat's profit when making and selling x rackets?

Solution

a) The total cost of producing 100 tennis rackets is
$f(100) = 150 + 75\sqrt{100} = 150 + 75 \cdot 10 = 150 + 750 = 900$.
Therefore, the average cost of producing 100 tennis rackets is $\frac{900}{100} = 9$ dollars per racket.

b) To find the marginal cost, first compute the derivative of the total-cost function. The total-cost function can be written
$f(x) = 150 + 75x^{\frac{1}{2}}$, so the derivative is $f'(x) = 75 \cdot \frac{1}{2}x^{-\frac{1}{2}}$.
Therefore, the marginal cost of producing the 100th racket is
$f'(100) = 75 \cdot \frac{1}{2}100^{-\frac{1}{2}} = 75 \cdot \frac{1}{2} \cdot \frac{1}{10} = 3.75$.

c) The total cost of producing x tennis rackets is $150 + 75\sqrt{x}$ and the total revenue from selling x rackets at \$7.50 each is $7.5x$. Pat's profits are the revenue minus the cost, or $7.5x - (150 + 75\sqrt{x})$.

The derivative can also be used by a business to determine what price they should sell their product for in order to maximize revenue.

Example

A minor-league hockey team sells game tickets for \$16 each. At this price, 3500 spectators attend each game. A study suggests that for every increase of 50 cents in ticket price, 100 fewer people will come to the game. How much should the team sell tickets for, in order to maximize their revenue?

Solution

You're told that an increase of 50 cents in the ticket price will result in the number of spectators decreasing by 100. That means if the ticket price is 16 + 0.50, the number of spectators will be 3500 – 100. If the ticket price is 16 + 0.50 + 0.50, then the number of spectators will be 3500 – 100 – 100, and so on. In general, if the ticket price is $16 + 0.50x$, the number of spectators will be $3500 - 100x$.

If you have $3500 - 100x$ spectators who each pay $16 + 0.50x$ dollars for a ticket, the total revenue is $(3500 - 100x)(16 + 0.50x)$ $= 56\ 000 + 1750x - 1600x - 50x^2 = 56\ 000 + 150x - 50x^2$.

The revenue function: $r(x) = 56\ 000 + 150x - 50x^2$

If you want the maximum of the total revenue, you need to find the derivative: $r'(x) = 150 - 100x$

As a result, the derivative is 0 when $x = \dfrac{150}{100} = 1.5$, the derivative is positive when $x < 1.5$, and the derivative is negative when $x > 1.5$. Therefore, the revenue is maximized when $x = 1.5$. An x-value of 1.5 means that the ticket price is $16 + 0.50(1.5) = 16.75$ (and the number of spectators is $3500 - 100(1.5) = 3350$).

Support Questions
(do not send in for evaluation)

1. A manufacturing company determines that the cost of manufacturing x items is $C(x) = 0.02x^2 + 40x + 5000$.
   a) What is the average cost of manufacturing 1000 items?
   b) What is the marginal cost of manufacturing the 1000th item?
2. A vendor sells T-shirts for \$12 each. At this price, she sells 200 per month. A study tells her that for every \$0.10 increase in the price of a T-shirt, there will be a reduction of 5 sales. At what price and how many T-shirts should she sell in order to maximize revenue?

There are Suggested Answers to Support Questions at the end of this unit.

Half-Life

Another example of a real-life situation where derivatives are relevant is calculating the half-life of a radioactive substance.

For example, suppose you have 1000 g of a radioactive substance at noon and you're told that its half-life is 1 hour. That means that every hour that goes by after noon, the amount of the substance gets multiplied by $\frac{1}{2}$. The amount of the radioactive substance is $1000\left(\frac{1}{2}\right)$ at 1 p.m., $1000\left(\frac{1}{2}\right)^2$ at 2 p.m., $1000\left(\frac{1}{2}\right)^3$ at 3 p.m., and so on. In general, when the time is t hours after noon, the total amount of radioactive substance is $1000\left(\frac{1}{2}\right)^t$ g.

Now imagine you have a radioactive substance with a half-life of 14.2 hours. How much radioactive substance will there be at time t? You know that 14.2 hours after noon the original amount will be multiplied by $\frac{1}{2}$. At 28.4 hours after noon, the original amount will be multiplied by $\left(\frac{1}{2}\right)^2$. In general, after n increments of 14.2 hours, the original amount will be multiplied by $\left(\frac{1}{2}\right)^n$. If the time is t hours after noon, then t hours is precisely a $\frac{t}{14.2}$ increment of 14.2 hours. Therefore, if the half-life is 14.2 hours, the amount of radioactive substance t hours after noon is $1000\left(\frac{1}{2}\right)^{\frac{t}{14.2}}$ g (if you start with 1000 g).

In general, if you begin with an initial amount A_0 of a radioactive substance whose half-life is k hours, then the amount after an elapsed time of t hours will be $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{k}}$.

To summarize:

Half-Life

The amount left after an elapsed time of t hours is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{k}}$, where k is the half-life and A_0 is the initial amount of radioactive substance.

Example

Suppose kryptonium is a radioactive substance with a half-life of 17 years. In 2008, you have a 1 kg sample of kryptonium. What will be the rate of decay in the year 2050?

Solution

The initial amount is 1 kg, and the half-life is 17 years. You can conclude that the amount of kryptonium t years after 2008 is given by $A(t) = 1\left(\frac{1}{2}\right)^{\frac{t}{17}}$ kg.

The question asks for the rate of change of this quantity; that is, the derivative. From your knowledge of the derivatives of exponential functions, together with the chain rule, you know that the derivative of $\left(\frac{1}{2}\right)^u$ is $\left(\frac{1}{2}\right)^u \cdot \ln\left(\frac{1}{2}\right) \cdot u'$. You can conclude that $A'(t) = \left(\frac{1}{2}\right)^{\frac{t}{17}} \cdot \ln\left(\frac{1}{2}\right) \cdot \frac{1}{17}$.

In the year 2050, you have $t = 42$. The rate of change at that time is $A'(42) = \left(\frac{1}{2}\right)^{\frac{42}{17}} \cdot \ln\left(\frac{1}{2}\right) \cdot \frac{1}{17} = -0.007356$ kg per year.

Notice the rate of change is negative, since the amount of kryptonium is decreasing.

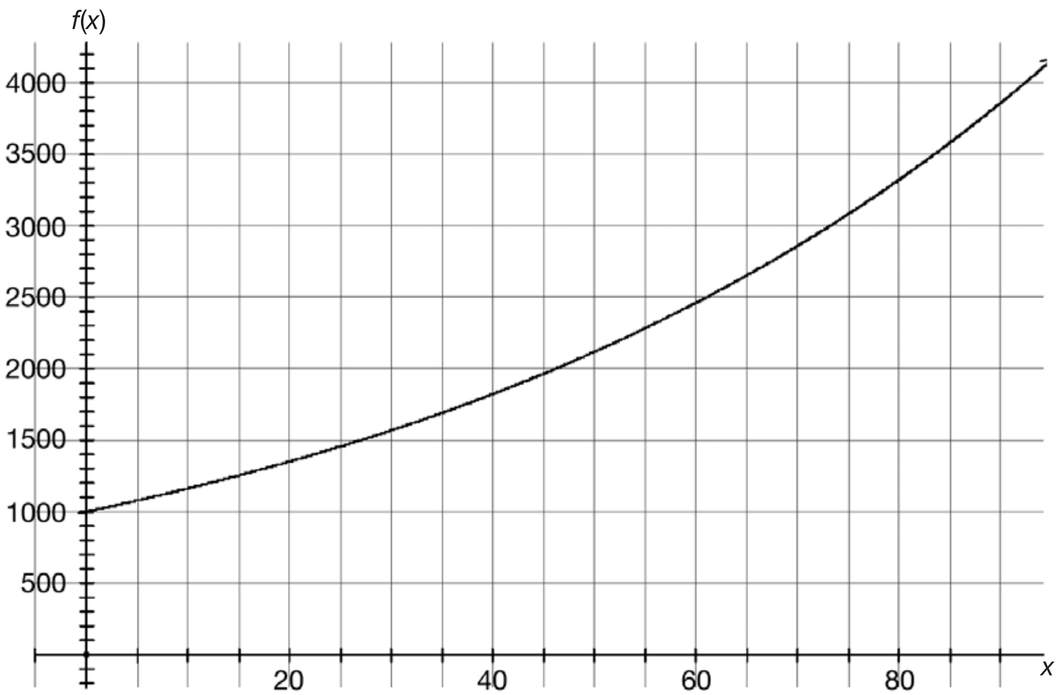
Support Question
(do not send in for evaluation)

3. Cesium-147 is a radioactive isotope with a half-life of 30 days. Suppose you start with a 500 g sample of Cesium-147. What will the rate of decay be after 40 days?

Population Growth

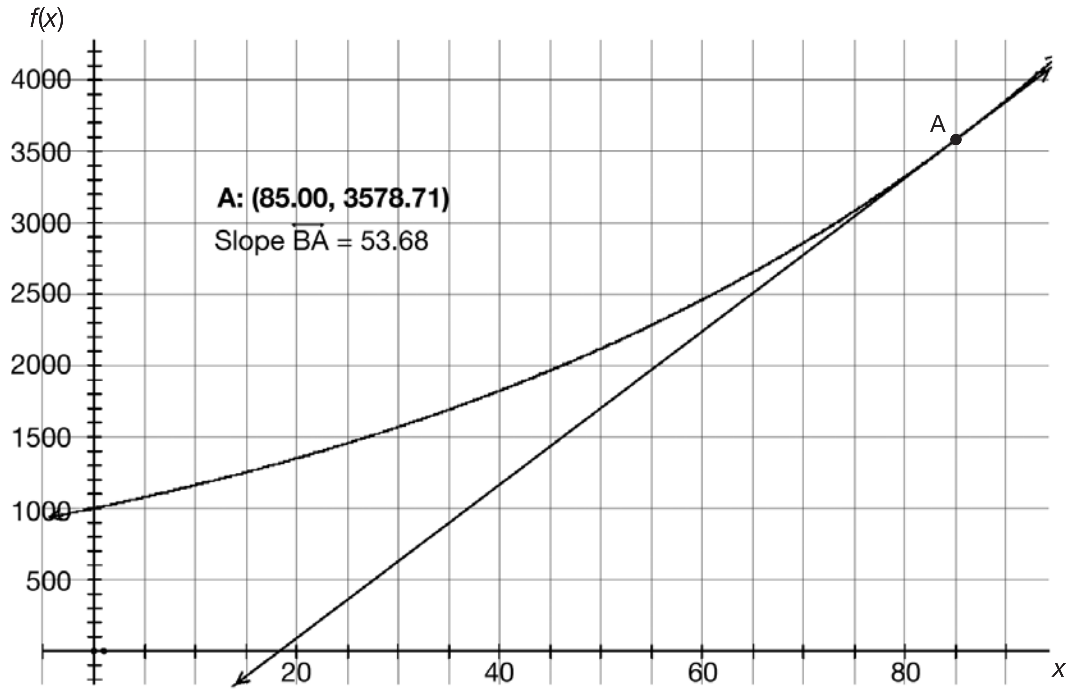
The study of population growth is another area where derivatives arise in practice and have a natural physical interpretation. Here is an example.

The graph shows the population of the fictional town of Ruritania as a function of x, where x is the number of years past 1900.



You can see that the population in 1900 was 1000, the population in 1920 was approximately 1300, and the population in 1960 was slightly below 2500.

Now consider the tangent to the curve at a particular point.



The y-coordinate at point A represents the population at that time. The slope of the tangent line, or equivalently, the derivative, represents the rate of growth of the population at that particular time. For example, you can see that in 1985 the population of Ruritania was almost 3600, and at that time the population was increasing at a rate of 53.68 people per year. Notice that the graph is less steep at earlier times, which means that with time the population is growing faster. For example, around 1920 the rate of growth was closer to 20 people per year.

Support Question
(do not send in for evaluation)

4. The population of butterflies is given by the function $P(t) = \dfrac{6000}{1 + 49(0.6)^t}$, where t is the time in days. Determine the rate of growth in the population after 5 days. Verify your answer graphically using the applet "Rate of Change Applications" on your course page at *ilc.org*.

ilc.org

Conclusion

In this lesson, you used your familiarity with derivatives to answer various problems involving marginal costs, half-life, and population growth. In Lesson 12, you will learn to solve more real-life problems using derivatives.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

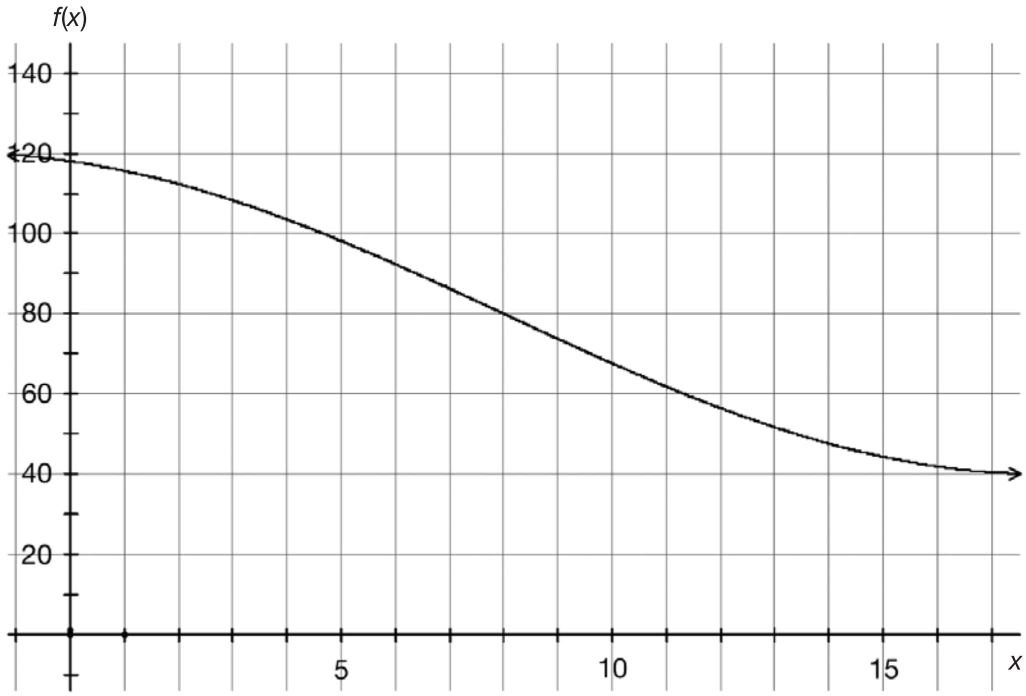
(18 marks)

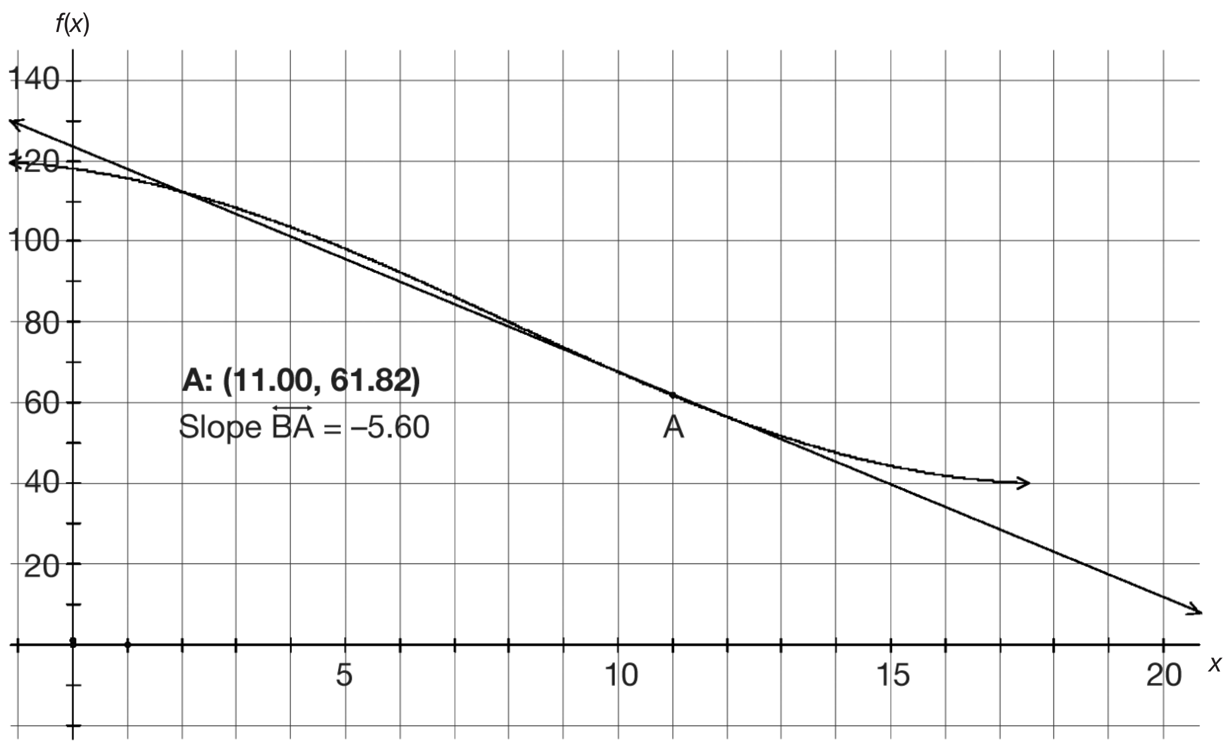
28. The first graph shows the size of a population of Florida panthers as a function of time, where x represents the number of years after 1980. The following two graphs show the same graph, including a tangent line at various points. **(6 marks: 2 marks each)**

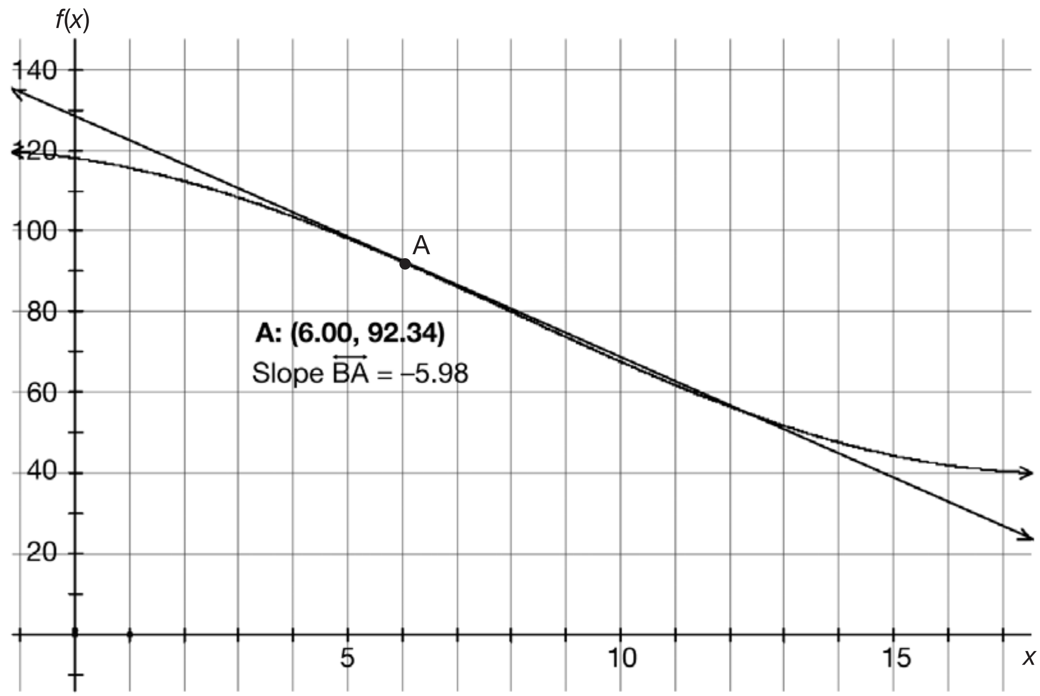
    a) Estimate the number of panthers in 1985 and 1996.

    b) At what rate was the population decreasing in 1986 and 1991?

    c) Estimate the time at which the panther population was decreasing the fastest.


f(x)
140
120
100
80
60
40
20
5
10
15
x


f(x)
140
120
100
80
60
40
20
A: (11.00, 61.82)
Slope $\overleftrightarrow{BA}$ = –5.60
A
5
10
15
20
x



29. After a particularly harsh winter, the regrowth of a population of rabbits in an area is given by the function $P(t) = \dfrac{500}{1 + e^{-t}}$ where t is the time in years. Determine the rate of growth in the population after 3 years. **(3 marks)**

30. a) The cost, in dollars, of producing x mattresses is $C(x) = 2500 + 100x - 0.1x^2$. Calculate the average cost of producing 200 mattresses and the marginal cost of producing the 200th mattress. **(4 marks)**

 b) A radioactive substance has a half-life of 20 days.

 i) How much time is required so that only $\dfrac{1}{32}$ of the original amount remains? **(3 marks)**

 ii) Find the rate of decay at this time. **(2 marks)**

Now go on to Lesson 12. Do not submit your coursework to ILC until you have completed Unit 3 (Lessons 11 to 15).

