MCV4U-A

10

Sketching Functions

Introduction

In this lesson, you will see how various techniques you have learned can be combined to help you sketch reasonable graphs of polynomial functions. You will rely heavily on the geometric interpretation of the first and second derivatives of a function.

Estimated Hours for Completing This Lesson	
Even and Odd Functions	0.5
Graphing Polynomial Functions	1.5
Graphing Rational Functions	1.5
Key Questions	1.5

What You Will Learn

After completing this lesson, you will be able to

- describe key features of a function if given information about its first or second derivative
- sketch a reasonable graph for a polynomial function by identifying symmetries, finding maximum or minimum points and intervals of increase or decrease, finding inflection points and intervals of concavity
- sketch a reasonable graph for a rational function by finding the domain, identifying symmetries, finding the asymptotes, finding maximum or minimum points and intervals of increase or decrease, finding inflection points and intervals of concavity

Even and Odd Functions

Some functions have certain types of symmetries that can serve as shortcuts if you want to draw their graphs. Specifically, there are two special types of functions that are known as even functions and odd functions.

Here are the definitions of even and odd functions. Functions will then be illustrated to show how identifying a function as even or odd can save you some time when graphing.

Even Functions and Odd Functions

The function $f(x)$ is called an even function if you have $f(-x) = f(x)$ for all x.

The function $f(x)$ is called an odd function if you have $f(-x) = -f(x)$ for all x.

The graph of an even function remains unchanged when you reflect it through the y-axis, and the graph of an odd function remains unchanged when you rotate it 180 degrees around the origin.

Here are a few examples to illustrate these concepts.

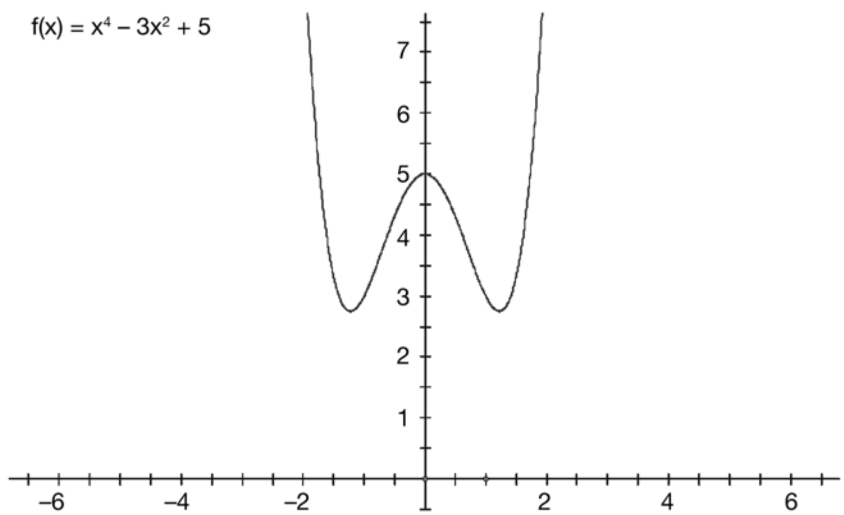
Suppose you want to sketch the graph of the polynomial function $f(x) = x^4 - 3x^2 + 5$. It would take some time to draw a very accurate graph by hand. This particular polynomial contains only even powers of x, however, which means that it behaves the same for negative values of x as for positive values of x. Algebraically, you can verify that $f(x)$ is an even function:

$$
\begin{aligned}
f(-x) &= (-x)^4 - 3(-x)^2 + 5 \\
&= x^4 - 3x^2 + 5 \\
&= f(x)
\end{aligned}
$$

When you replace each x in the definition of $f(x)$ with $-x$ and then simplify, you get the original function $f(x)$. That is, $f(-x) = f(x)$, so $f(x)$ is an even function.

How does this serve as a shortcut when graphing the function? The fact that $f(-x) = f(x)$ for all x means essentially that the right half of the graph looks the same as the left half. When drawing the graph by hand, you can sketch the graph for positive values of x only, and for negative values of x you simply draw the reflection of what you have already drawn.

To illustrate, here's the graph of the function $f(x) = x^4 - 3x^2 + 5$, which is discussed earlier:



It's worth mentioning here that most functions are neither even nor odd. Don't be misled by the terms "even" and "odd." It's true that when you're talking about integer numbers {–K, –3, –2, –1, 0, 1, 2, 3, K}, every number is either even or odd, but this is *not* true for functions.

If you have a function $f(x)$ and consider $f(-x)$, three things can happen:

- $f(-x)$ turns out to be the same as $f(x)$
- $f(-x)$ turns out to be the same as $-f(x)$
- $f(-x)$ turns out to be nothing special: not the original function, and not the negative of the original function

Consider the function $g(x) = 12x^3 - 5x^2 + 4x + 8$. What happens in this case when you consider $g(-x)$?

$$g(-x) = 12(-x)^3 - 5(-x)^2 + 4(-x) + 8$$
$$= -12x^3 - 5x^2 - 4x + 8.$$

You don't get exactly $g(x)$, and you also don't get $-g(x)$. Therefore, $g(x)$ is neither even nor odd.

Notice that when you replaced x with $-x$ in this case, the terms $12x^3$ and $4x$ were replaced with their negatives, whereas the terms $-5x^2$ and $+8$ were unchanged. In general, when you replace x with $-x$ in a polynomial, every odd power of x will be replaced with its negative, and every even power of x will be unchanged. Notice that the constant term $+8$ counts as an even power of x, since it's the same as $+8x^0$.

Here's a summary of these observations:

Even and Odd Polynomial Functions

Suppose $f(x)$ is a polynomial function:

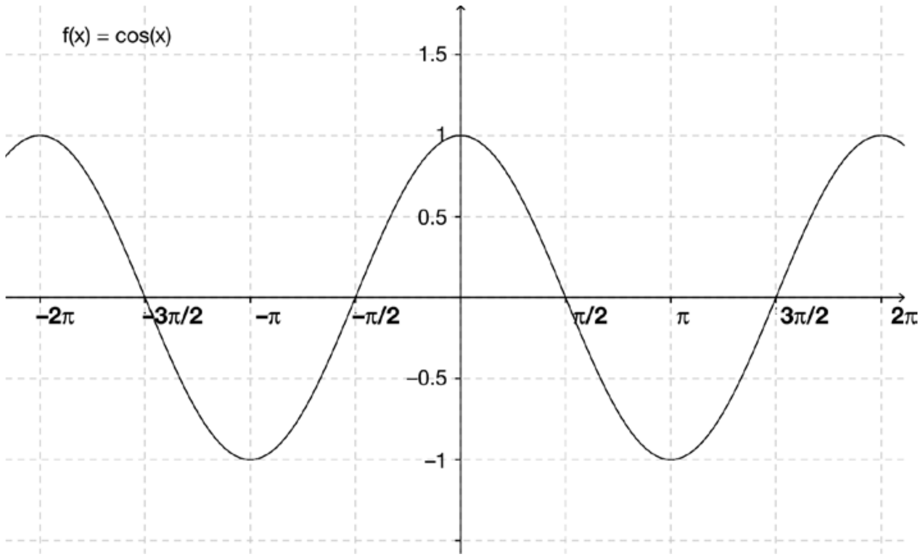
If $f(x)$ contains only even powers of x, then $f(x)$ is an even function.

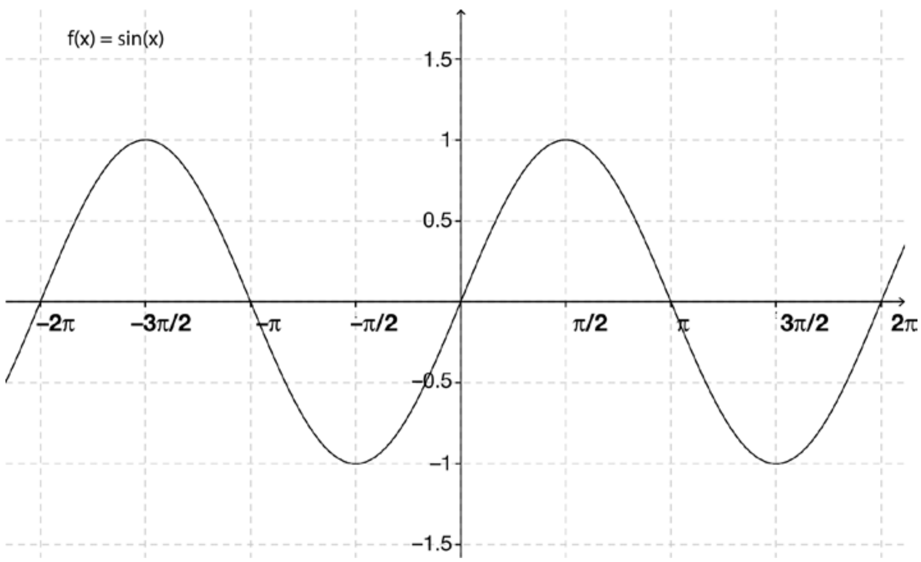
If $f(x)$ contains only odd powers of x, then $f(x)$ is an odd function.

If $f(x)$ contains both even and odd powers of x, then $f(x)$ is neither an even function nor an odd function.

A constant term, if the polynomial has any, counts as an even power of x.

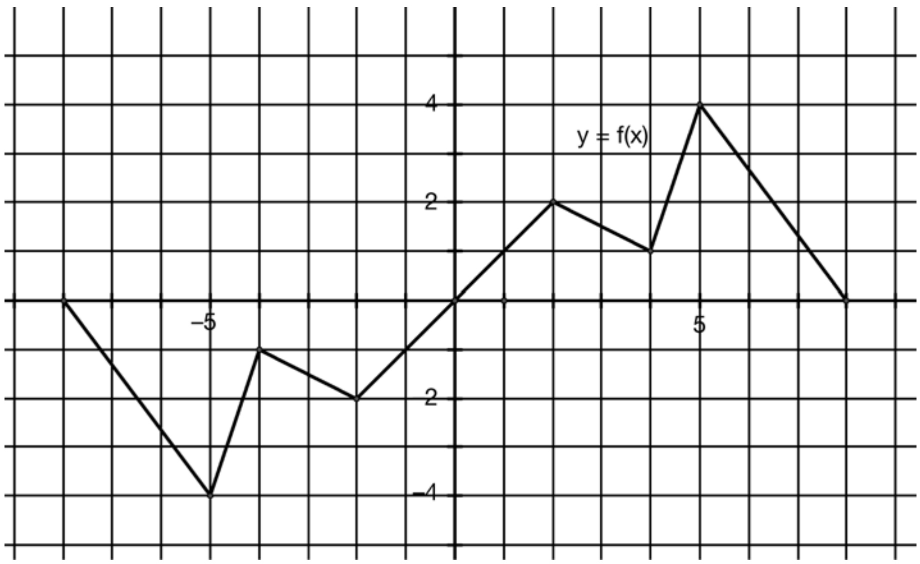
The concepts of even function and odd function also apply to functions that aren't polynomials. For example, the cosine function is even and the sine function is odd. This can be illustrated with the familiar graphs:





You can also see that the graph of the cosine function looks the same if reflected through the y-axis, or equivalently, that $\cos(-x) = \cos x$, so $\cos x$ is an even function. You can see that the graph of the sine function looks the same if rotated 180 degrees around the origin or, equivalently, that $\sin(-x) = -\sin x$, so $\sin x$ is an odd function.

As another example, consider the function shown in the following graph:



In this picture, even though you aren't given a formula for $f(x)$, you can say that $f(x)$ is an odd function because its graph looks the same if you rotate it 180 degrees around the origin. Notice also that you can verify that $f(-x) = -f(x)$ at particular points. For example: $f(4) = 1$ and $f(-4) = -1$, $f(5) = 4$ and $f(-5) = -4$.

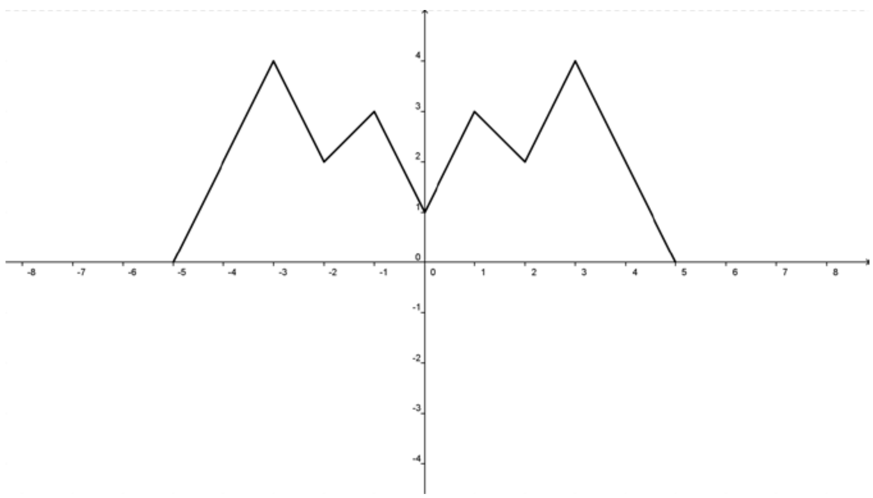
Support Question
(do not send in for evaluation)

17. Determine if the following functions are even, odd, or neither:

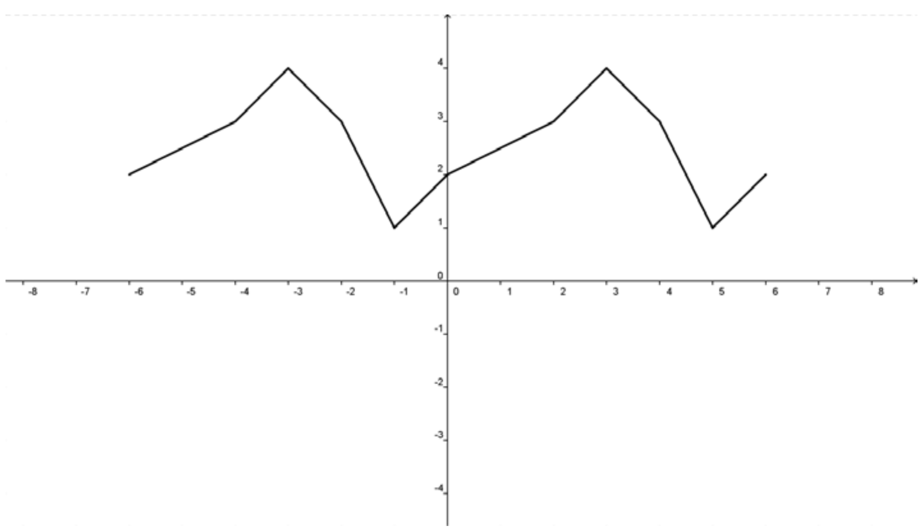
    a) $f(x) = x^3 - 5x$

    b) $g(x) = x^4 - 3$

c)



d)



There are Suggested Answers to Support Questions at the end of this unit.

Graphing Polynomial Functions

You're now going to see how some of the things you've recently learned about derivatives and second derivatives can be combined to help you sketch graphs of polynomial functions.

Example

Sketch on graphing paper a reasonably accurate graph of the function $f(x) = x^3 - 9x^2 + 15x - 7$.

Solution

Both the first and second derivative of $f(x)$ will be useful for determining information about the graph. Before doing that, you might be able to determine some useful information from the original function $f(x)$ itself; namely, information about symmetry and intercepts.

Using symmetry simply means to check if $f(x)$ is an odd or an even function, or neither. In this case, $f(x)$ is a polynomial containing both odd and even powers of x, so $f(x)$ is neither odd nor even, and there are no obvious symmetries to take advantage of.

When sketching a graph of $f(x)$, before even considering the first or second derivative, you can find particular points on the graph simply by substituting numbers into the original function $f(x)$. This would be time-consuming and unstructured if you just did it blindly using random numbers. Instead, you can focus on special points, known as the intercepts, of the graph. You may remember that the x-intercepts are the points where the graph crosses the x-axis (the points where $y = 0$), and the y-intercepts are the points where the graph crosses the y-axis (the points where $x = 0$).

Find the y-intercepts by plugging $x = 0$ into the function $f(x)$. You'll find that $f(0) = 0^3 - 9(0)^2 - 15(0) - 7 = -7$, so the y-intercept is the point $(0, -7)$. As for the x-intercepts, you find them by setting y equal to 0, which means you have to solve the equation $x^3 - 9x^2 + 15x - 7 = 0$.

Normally, a degree-3 polynomial like this is difficult to solve, but you may be able to find one root by trial and error. If you experiment with simple values of x, you soon notice that when you substitute $x = 1$, you find that $f(1) = 0$. This means that $x - 1$ must be a factor of the polynomial. When you divide $x^3 - 9x^2 + 15x - 7$ by $x - 1$ using long division, you find that $f(x)$ factors as $(x - 1)$ times $(x^2 - 8x + 7)$. In this case, you are fortunate that $x^2 - 8x + 7$ happens to factor further.

You can conclude that $f(x)$ factors as $x^3 - 9x^2 + 15x - 7 = (x - 1)(x - 1)(x - 7)$. (**Note**: If you do not see the factors of a quadratic equation, you should use the quadratic formula to find the roots.)

To conclude, you have $y = 0$ if $x = 1$ or $x = 7$, so the x-intercepts are the points $(1, 0)$ and $(7, 0)$.

Next, you should focus on the information that can be obtained from the derivative.

From the rules, you know for differentiating polynomials that you can say $f'(x) = 3x^2 - 18x + 15$ and $f''(x) = 6x - 18$.

Finding the zeros of the derivative enables you to tell when the derivative is positive and when the derivative is negative. This helps you identify the intervals of increase and decrease of $f(x)$ as well as its extreme points.

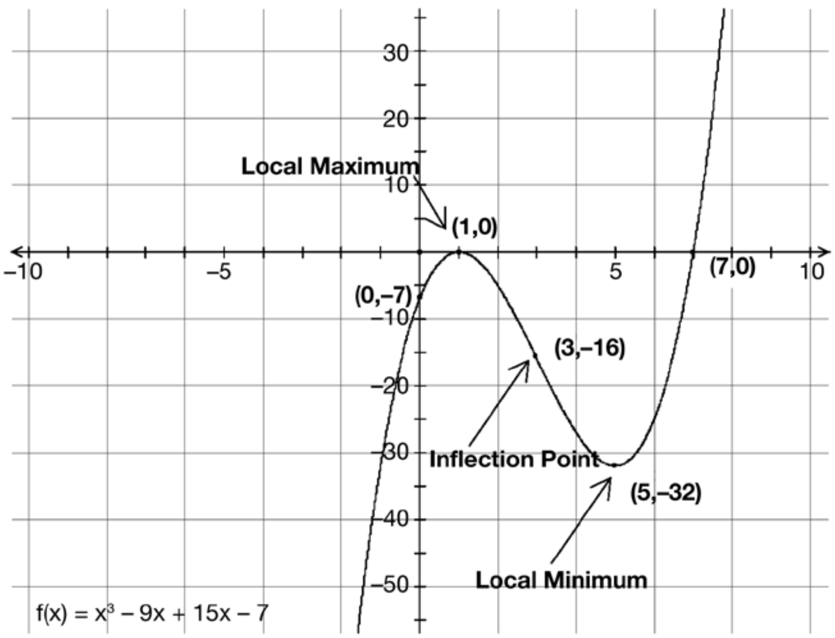
Find the zeros of $f'(x)$ by factoring:

$$\begin{aligned} f'(x) &= 3x^2 - 18x + 15 \\ &= 3(x^2 - 6x + 5) \\ &= 3(x - 1)(x - 5) \end{aligned}$$

Thus, $f'(x) = 0$ when $x = 1$ or $x = 5$. You'll find that $f'(x)$ is positive when $x < 1$, $f'(x)$ is negative when $1 < x < 5$, and $f'(x)$ is positive again when $x > 5$. You can conclude that $f(x)$ is increasing when $x < 1$, $f(x)$ is decreasing when $1 < x < 5$, and $f(x)$ is increasing again when $x > 5$. You can then also say that $f(x)$ has a maximum at $x = 1$ and $f(x)$ has a minimum at $x = 5$.

Next, focus on the second derivative $f''(x) = 6x - 18$ and see what it can tell you. The second derivative factors easily as $f''(x) = 6(x - 3)$. You can conclude that $f''(x) = 0$ when $x = 3$, that $f''(x)$ is negative when $x < 3$, and that $f''(x)$ is positive when $x > 3$. Therefore, the graph of $f(x)$ is concave down when $x < 3$ and is concave up when $x > 3$, and there is an inflection point at $x = 3$.

All of the information you've obtained can now be combined into a graph of $f(x)$:



Example

Sketch a reasonably accurate graph of the function $f(x) = x^3 - 11x^2 + 24x$.

Solution

Symmetry: Since $f(x)$ is a polynomial containing both even and odd powers of x, $f(x)$ is neither an even function nor an odd function. Therefore, the function has no obvious symmetry.

Intercepts: Find the y-intercept by substituting $x = 0$. You'll find that $f(0) = 0$, so the y-intercept is the point (0, 0). Find x-intercepts by solving $f(x) = 0$:

$x^3 - 11x^2 + 24x = 0$

$x(x^2 - 11x + 24) = 0$

$x(x - 3)(x - 8) = 0$

$f(x) = 0$ when $x = 0$, 3, or 8, so the x-intercepts are the points (0, 0), (3, 0), and (8, 0).

First derivative: The first derivative is $f'(x) = 3x^2 - 22x + 24$. You can find where $f'(x)$ is positive or negative by first finding where $f'(x) = 0$. It's possible to solve $3x^2 - 22x + 24 = 0$ by factoring, although since the coefficients are somewhat large, you may prefer to use the quadratic formula. Here are the solutions to $3x^2 - 22x + 24 = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-22) \pm \sqrt{(-22)^2 - 4(3)(24)}}{2(3)}$$

$$x = \frac{22 \pm \sqrt{484 - 288}}{6}$$

$$x = \frac{22 \pm \sqrt{196}}{6}$$

$$x = \frac{22 \pm 14}{6}$$

The solutions to $f'(x) = 0$ are $x = \frac{36}{6} = 6$ and $x = \frac{8}{6} = \frac{4}{3} \approx 1.333$. Once you know that $x = 6$ is a root of $3x^2 - 22x + 24$, you know that the intervals that you need to look at are $x < \frac{4}{3}$, $\frac{4}{3} < x < 6$, and $x > 6$. Now it is easy to show that $f'(x)$ is positive when $x < \frac{4}{3}$, $f'(x)$ is negative when $\frac{4}{3} < x < 6$, and $f'(x)$ is positive again when $x > 6$.

Intervals of increase or decrease: Now that you've found where $f'(x)$ is positive and negative, you can say that $f(x)$ is increasing when $x < \frac{4}{3}$, $f(x)$ is decreasing when $\frac{4}{3} < x < 6$, and $f(x)$ is increasing again when $x > 6$.

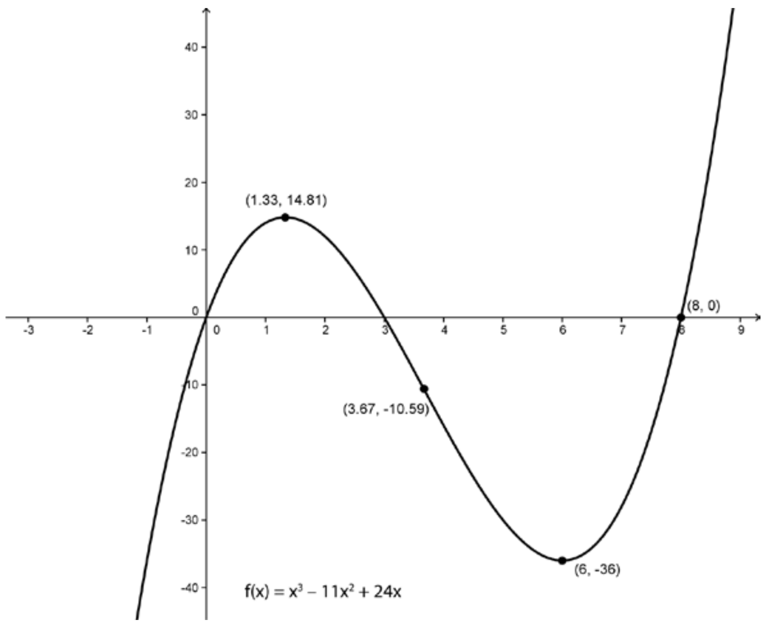
Extreme points: Now that you know where $f(x)$ is increasing and decreasing, you can say that $f(x)$ has a maximum at $x = \frac{4}{3}$ and a minimum at $x = 6$.

Second derivative: The second derivative is $f''(x) = 6x - 22$. You'll quickly find that the only zero of the second derivative is $x = \frac{22}{6} = \frac{11}{3} \approx 3.667$. Therefore, $f''(x)$ is negative when $x < \frac{11}{3}$ and $f''(x)$ is positive when $x > \frac{11}{3}$.

Concavity: Based on the second derivative, you can say that $f(x)$ is concave down when $x < \frac{11}{3}$, and $f(x)$ is concave up when $x > \frac{11}{3}$.

Inflection points: The only inflection point is at $x = \frac{11}{3}$, since that is the only point where the graph has a change in concavity.

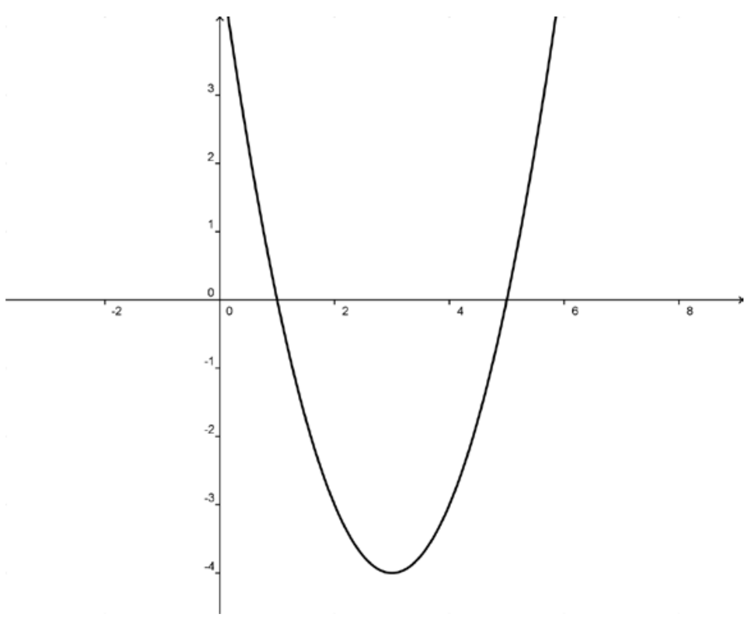
Graph: Combine all the above information into a graph of $f(x)$:



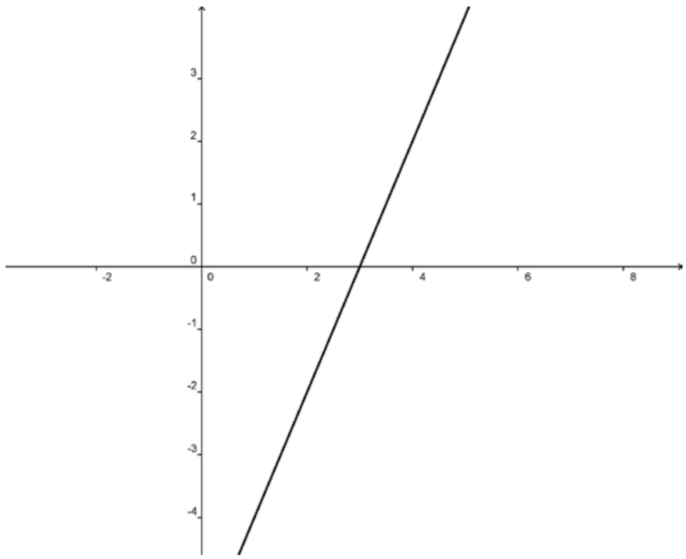
Example

The graphs of $f'(x)$ and $f''(x)$ are given. Suppose you're also told that $f(x)$ is zero at $x = 1$ and $x = 7$. Use this information to sketch $f(x)$.

Graph of $f'(x)$:



Graph of $f''(x)$:



Solution

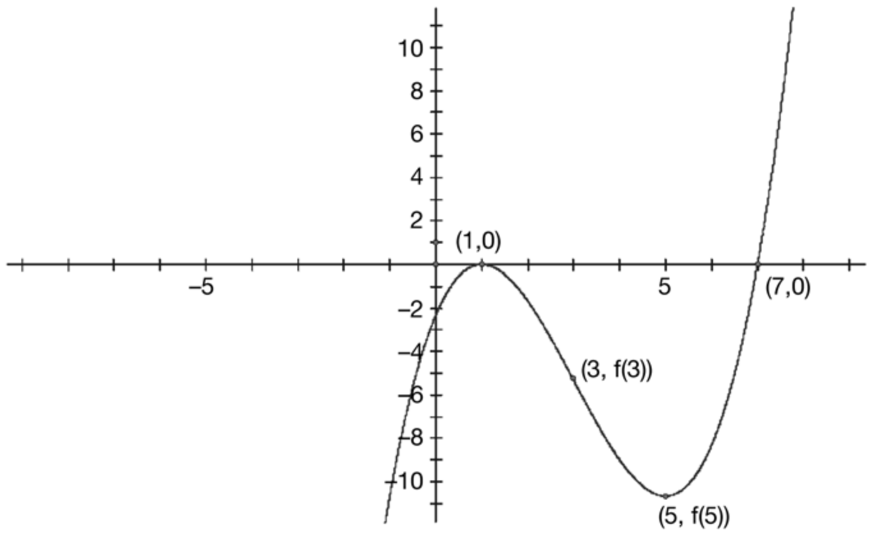
The answers to these questions will give you the important information:

- Where is $f'(x)$ positive, negative, or zero?
- Where is $f''(x)$ positive, negative, or zero?

You can see from the first graph that $f'(x)$ is positive when $x < 1$, $f'(x)$ is negative when $1 < x < 5$, and $f'(x)$ is positive again when $x > 5$. Therefore, $f(x)$ must be increasing when $x < 1$, decreasing when $1 < x < 5$, and increasing again when $x > 5$. This means $f(x)$ must have a maximum at $x = 1$ and a minimum at $x = 5$.

You can also see that $f''(x)$ is negative when $x < 3$ and is positive when $x > 3$. Therefore, $f(x)$ must be concave down when $x < 3$ and concave up when $x > 3$. This means $x = 3$ is an inflection point.

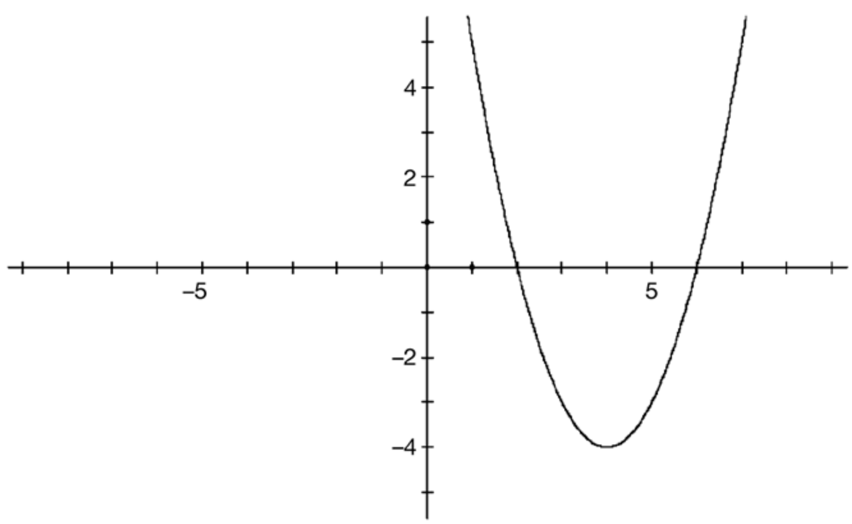
You can now construct the graph of $f(x)$ by plotting the given intercept points (1, 0) and (7, 0) and then drawing $f(x)$ to be increasing or decreasing, and concave up or concave down, on the appropriate intervals. Since you weren't given explicit formulas for $f'(x)$ or $f''(x)$, you don't have enough information to find exact values of $f(x)$ at points other than the given points of (1, 0) and (7, 0). Nonetheless, you can describe the correct overall shape of the graph.



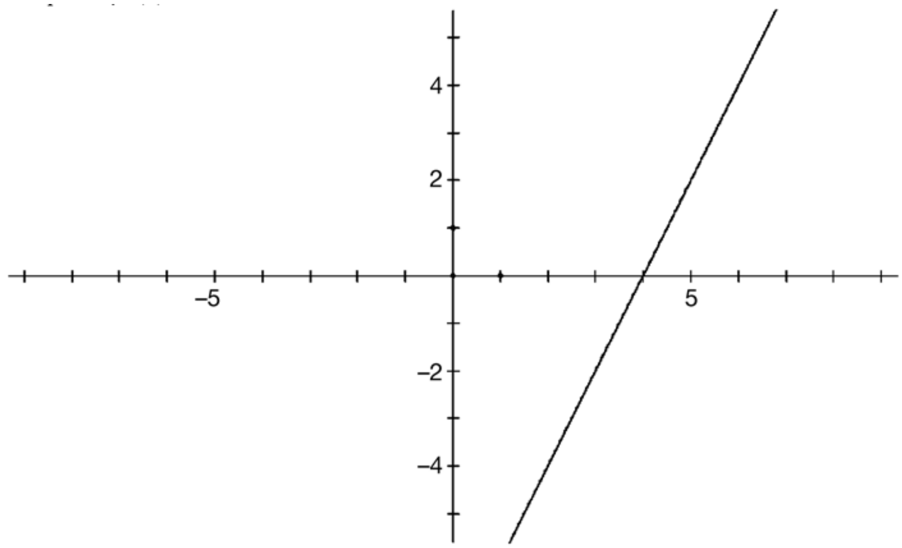
Support Questions
(do not send in for evaluation)

18. The graphs of $f'(x)$ and $f''(x)$ are given. Suppose you're also told that $f(x)$ has zeros at $x = 2$ and $x = 8$. Use this information to sketch $f(x)$.

    Graph of $f'(x)$:



Graph of $f''(x)$:



19. Sketch the function $f(x) = x^3 - 3x^2 - x + 3$.

Graphing Rational Functions

In this section, you will see a similar approach to graphing rational functions. Recall that a rational function has the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials. For example,

$y = \frac{2x-3}{x^2+1}$ is a rational function.

Example

Sketch on graphing paper a reasonably accurate graph of the function $f(x) = \frac{x-2}{x-1}$.

Solution

Domain: You should start by finding the domain of the function. Recall that the domain of a function is the set of all possible values of x that can be substituted in the function.

Since $f(x)$ is a rational function, the denominator cannot be zero, so $x - 1 \neq 0$ and hence $x \neq 1$. You can conclude that the domain is all real numbers except $x = 1$.

Asymptotes: Recall from the Advanced Functions course (MHF4U) that an asymptote to a function is a line that a function approaches but does not become identical to.

You can find the vertical asymptote of $f(x)$ by finding $\lim_{x\to1} \frac{x-2}{x-1}$ and the horizontal asymptotes by finding $\lim_{x\to\infty} \frac{x-2}{x-1}$:

$$\lim_{x\to1^+} \frac{x-2}{x-1} = -\infty$$

$$\lim_{x\to1^-} \frac{x-2}{x-1} = +\infty$$

The line $x = 1$ is a vertical asymptote.

$$\lim_{x\to\infty}\frac{x-2}{x-1}=\lim_{x\to\infty}\frac{x\left(1-\frac{2}{x}\right)}{x\left(1-\frac{1}{x}\right)}$$

As $x \to \infty$, both $\frac{2}{x}$ and $\frac{1}{x}$ go to zero. Therefore,

$$\lim_{x\to\infty}\frac{\left(1-\frac{2}{x}\right)}{\left(1-\frac{1}{x}\right)}=1$$

The line $y = 1$ is a horizontal asymptote.

Symmetry: Since $f(-x)=\frac{-x-2}{-x-1}$, $f(x)$ is neither even nor odd. Therefore, the function has no obvious symmetry.

Intercepts:

x-intercepts: $\frac{x-2}{x-1}=0$ when the numerator is zero

$x - 2 = 0$

$x = 2$

(2, 0) is the x-intercept.

y-intercept:

$$\frac{0-2}{0-1}=2.$$

(0, 2) is the y-intercept.

Both the first and second derivatives of $f(x)$ will be useful for determining information about the graph.

First derivative:

To find the derivative, start by writing $\frac{x-2}{x-1} = (x-2)(x-1)^{-1}$ and applying the product rule:

$$f'(x) = (x-2)(-1)(x-1)^{-2} + (x-1)^{-1}$$

$$= \frac{-(x-2)}{(x-1)^2} + \frac{1}{(x-1)} = \frac{-(x-2)+(x-1)}{(x-1)^2} = \frac{1}{(x-1)^2}$$

$f'(x)$ is never zero, so the function has no local maximums or minimums.

Extreme points:

$f'(x) > 0$ for all x, since $(x-1)^2$ is always positive. You can conclude that the function is always increasing.

Second derivative:

To find the second derivative, write $f'(x)$ as a power:

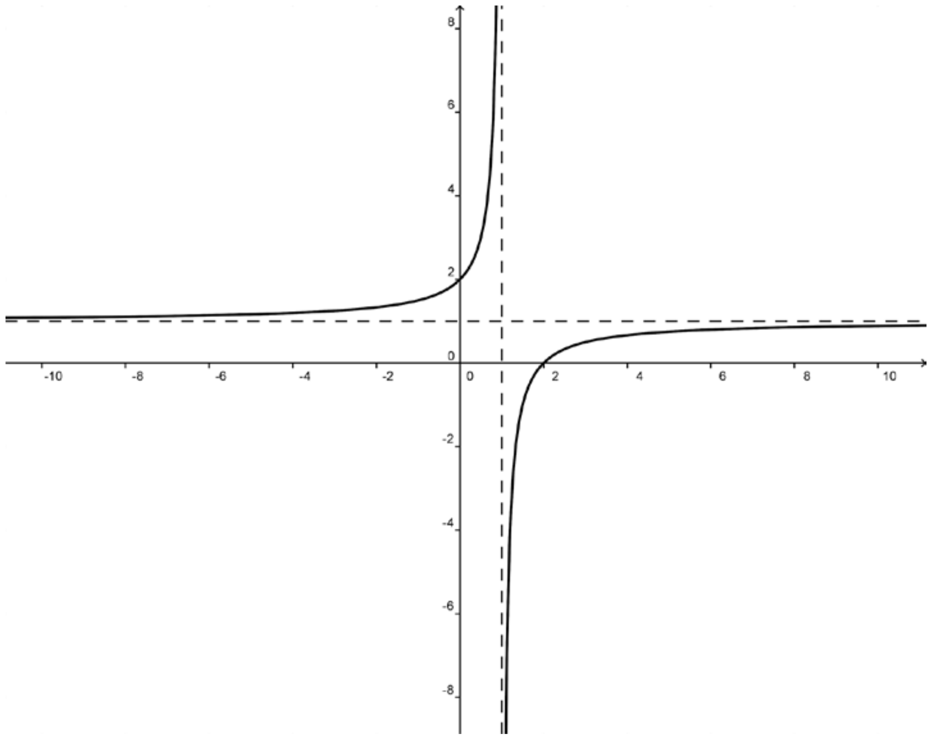
$f'(x) = (x-1)^{-2}$

$f''(x) = -2(x-1)^{-3}$

$f''(x)$ has no zeros

Concavity: $f''(x) > 0$ and $f(x)$ concave up when $x < 1$, and $f''(x) < 0$ and $f(x)$ concave down when $x > 1$.

Graph: Start the graph by drawing the asymptotes, the points of intersection with the x-axis and y-axis, and the function itself using the information in this solution.



Example

Sketch on graphing paper a reasonably accurate graph of the following function without studying the concavity or finding the second derivative:

$$f(x) = \frac{x^2 - 4}{x^2 - 1}$$

Solution

Domain: Start by finding the zeros of the denominator,

$x^2 - 1 = (x - 1)(x + 1) = 0$ so $x = 1$ or $x = -1$. Therefore, the domain is all real numbers except $x = -1$ and $x = 1$.

Vertical asymptotes:

$$\lim_{x \to -1^-} \frac{x^2 - 4}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{x^2 - 4}{x^2 - 1} = +\infty$$

$$\lim_{x \to 1^-} \frac{x^2 - 4}{x^2 - 1} = +\infty$$

$$\lim_{x \to 1^+} \frac{x^2 - 4}{x^2 - 1} = -\infty$$

$x = -1$ and $x = 1$ are two vertical asymptotes.

Horizontal asymptotes:

$$\lim_{x \to \infty} \frac{x^2 - 4}{x^2 - 1} = \lim_{x \to \infty} \frac{x^2\left(1 - \frac{4}{x^2}\right)}{x^2\left(1 - \frac{1}{x^2}\right)} = 1$$

$y = 1$ is a horizontal asymptote.

Intercepts:

$f(x) = 0$ when the numerator is 0

$x^2 - 4 = 0$

$(x - 2)(x + 2) = 0$ so $x = 2$ and $x = -2$

The x-intercepts are $(-2, 0)$, and $(2, 0)$.

$f(0) = 4$ so $(0, 4)$ is the y-intercept.

First derivative: Write $f(x)$ as a product and differentiate using the product rule:

$$f(x) = \frac{x^2 - 4}{x^2 - 1} = (x^2 - 4)(x^2 - 1)^{-1}$$

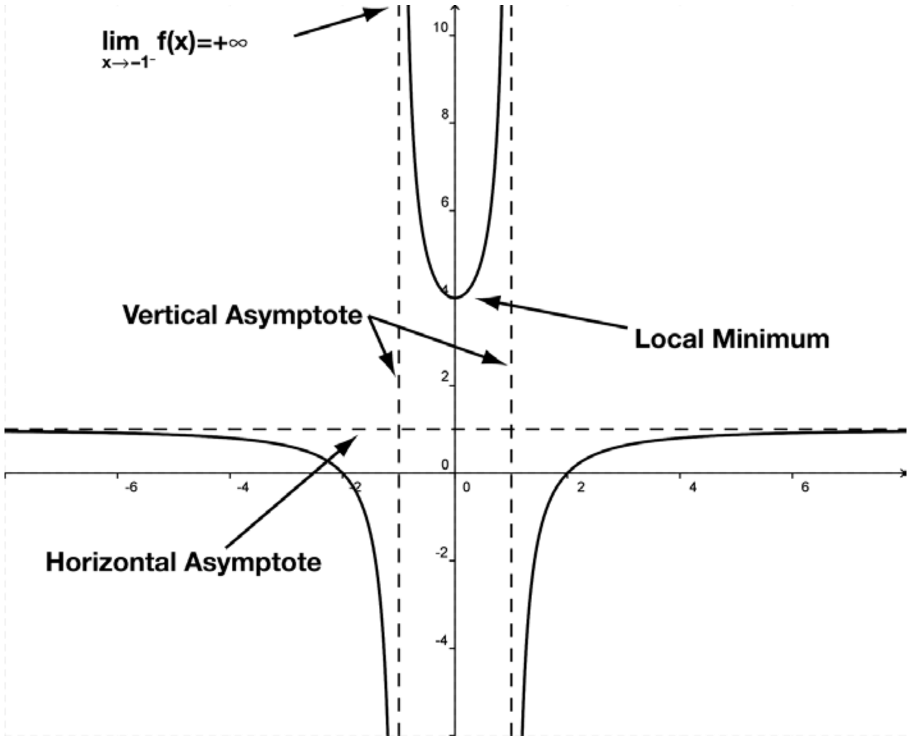
$$\begin{aligned} f'(x) &= -(2x)(x^2 - 4)(x^2 - 1)^{-2} + 2x(x^2 - 1)^{-1} \\ &= \frac{-(2x)(x^2 - 4)}{(x^2 - 1)^2} + \frac{2x}{(x^2 - 1)} \\ &= \frac{-(2x)(x^2 - 4) + 2x(x^2 - 1)}{(x^2 - 1)^2} \\ &= \frac{-2x^3 + 8x + 2x^3 - 2x}{(x^2 - 1)^2} \\ &= \frac{6x}{(x^2 - 1)^2} \end{aligned}$$

$f'(x) = 0$ when $x = 0$, $f'(x) > 0$ when $x > 0$ and $f'(x) < 0$ when $x < 0$.

Intervals of increase or decrease: The function is decreasing when $x < 0$ and increasing when $x > 0$.

Extreme points: $(0, 4)$ is a local minimum.

Graph:



Support Questions
(do not send in for evaluation)

20. Given $y = -3x^2 + 4x - 1$:

    a) State whether the function is odd or even.

    b) Find the x- and y-intercepts.

    c) Find the first and second derivatives, the intervals on which the function is increasing and decreasing, inflection points, and concavity.

    d) Sketch the function.

21. Given $f(x) = \dfrac{2x - 6}{x + 1}$:

a) State the domain.

b) Find the vertical and horizontal asymptotes.

c) Find the x- and y-intercepts.

d) Calculate the first and second derivatives, the intervals on which the function is increasing and decreasing, inflection points and concavity.

e) Sketch the function.

Conclusion

In this lesson, you have seen how the properties you learned about derivatives come together to help you sketch reasonable graphs of polynomial functions. By now you are familiar with the concept of the derivative as a rate of change, and you have spent much time learning how to compute the derivatives of various functions. In the first two lessons of Unit 3 (Lessons 11 and 12), you will apply what you learned to answer various real-life problems that involve rates of change.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(35 marks)

25. A function has a local maximum at $x = -2$, a local maximum at $x = 6$, and a local minimum at $x = 1$. What does this information tell you about the function? Sketch a possible function that has these characteristics. **(4 marks)**

26. For the function $g(x) = (1 - 2x)^2(x - 3)$

    i) State whether the function is odd or even. **(1 mark)**

    ii) Find the x- and y-intercepts. **(2 marks)**

    iii) Find the first and second derivatives, the intervals on which the function is increasing and decreasing, inflection points and concavity. **(11 marks)**

    iv) Sketch the function. **(3 marks)**

27. Given $f(x) = \dfrac{x + 5}{2x - 4}$:

    a) State the domain. **(1 mark)**

    b) Find the vertical and horizontal asymptotes. **(4 marks)**

    c) Find the x- and y-intercepts. **(1 mark)**

    d) Calculate the first and second derivatives, the intervals on which the function is increasing and decreasing, inflection points and concavity. **(5 marks)**

    e) Sketch the function. **(3 marks)**

This is the last lesson in Unit 2. When you are finished, do the Reflection for Unit 2. Follow any other instructions you have received from ILC about submitting our coursework, then send it to ILC. A teacher will mark your work and you will receive your results online.