MCV4U-A

9

Applications of the Derivative: Key Features of a Polynomial Function

Introduction

In Lesson 8, you learned about the relationship between the derivative of a function and the function itself. In this lesson, you will start with the algebraic equation of a polynomial function and graph it using the connection you studied in previous lessons.

Estimated Hours for Completing This Lesson	
Key Features of Functions	3.5
Key Questions	1.5

What You Will Learn

After completing this lesson, you will be able to

- describe when a function is increasing or decreasing by algebraically analyzing its derivative
- determine the concavity of a function by algebraically analyzing its second derivative

Key Features of Functions

In previous lessons, you learned how the second derivative of a function helps you learn about the behaviour of the function itself. You also learned that if the derivative is positive in an interval, the function is increasing, and that a negative derivative shows that the function is decreasing. You also learned about the second derivative of a function and how it gives you information about the concavity of a function. In this lesson, you will study the behaviour of a function starting from its equation.

Examples

Find the local maximum and minimum values of each function:

a) $f(x) = x^2 - 3x + 4$

b) $f(x) = 4x^3 - 6x^2 - 2$

c) $g(x) = \dfrac{x^2}{x - 4}$

Solutions

a) Start by calculating the first and second derivative of $f(x)$:

$f'(x) = 2x - 3$

$f''(x) = 2$

Then find the value of x when the first derivative equals zero.

$f'(x) = 0$

$2x - 3 = 0$

$2x = 3$

$x = \dfrac{3}{2}$

$$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 4$$

$$= \frac{9 - 18 + 16}{4}$$

$$= \frac{7}{4}$$

The point $\left(\frac{3}{2},\frac{7}{4}\right)$ is a critical point of $f(x)$.

$f'(x) > 0$ when $x > \frac{3}{2}$, so the function is increasing when $x > \frac{3}{2}$; $f'(x) < 0$ when $x < \frac{3}{2}$, so the function is decreasing when $x < \frac{3}{2}$.

You have enough information to assert that the point $\left(\frac{3}{2},\frac{7}{4}\right)$ is a local minimum, since the function is decreasing when you approach $\frac{3}{2}$ from the left and increasing when you approach $\frac{3}{2}$ from the right. You can confirm this by calculating the second derivative at $x = \frac{3}{2}$ and finding the concavity of the function. $f''(x) = 2 > 0$ for all x. The second derivative is always larger than zero, and $f''\left(\frac{3}{2}\right) > 0$, so the function is concave up at $x = \frac{3}{2}$. This confirms that $\left(\frac{3}{2},\frac{7}{4}\right)$ is a local minimum.

b) $f(x) = 4x^3 - 6x^2 - 2$

$f'(x) = 12x^2 - 12x$

$f'(x) = 12x(x - 1)$

$f'(x) = 0$ when $x = 0$ or when $x = 1$. The points $(0, f(0))$ and $(1, f(1))$ are critical points of the function.

To check when the derivative is positive and negative, you need to check the value of the derivative for values of x in each of the following intervals: $x < 0$, $0 < x < 1$, and $x > 1$.

$f'(-2) = 12(-2)^2 - 12(-2)$

$= 72 > 0$. The derivative is positive when $x < 0$ and the function is increasing.

$f'(\frac{1}{2}) = 12(\frac{1}{4}) - 12(\frac{1}{2})$

$= 3 - 6$

$= -3 < 0$. The derivative is negative when $0 < x < 1$ and the function is decreasing.

$f'(2) = 12(2)^2 - 12(2)$

$= 24 > 0$ The derivative is positive when $x > 1$ and the function is increasing.

You have enough information to assert that $(0, -2)$ is a local maximum and $(1, -4)$ is a local minimum. You can confirm this by determining the concavity of the function using the second derivative of $f(x)$. The second derivative is the derivative of the first derivative, $f'(x) = 12x^2 - 12x$:

$f''(x) = 24x - 12$

$f''(x) = 0$ when $x = \frac{1}{2}$, $f''(x) < 0$ when $x < \frac{1}{2}$ and $f''(x) > 0$ when $x > \frac{1}{2}$. Therefore, the function is concave down when $x < \frac{1}{2}$ and concave up when $x > \frac{1}{2}$. In particular, $f(x)$ is concave down at $x = 0$ and concave up at $x = 1$.

You can conclude that $(0, -2)$ is a local maximum and $(1, -4)$ is a local minimum.

c) $g(x) = \dfrac{x^2}{x-4}$

Rewrite $g(x)$ as $x^2(x-4)^{-1}$ and use the product rule to find the derivative:

$$
\begin{aligned}
g'(x) &= (x^2)'(x-4)^{-1} + x^2((x-4)^{-1})' \\
&= (2x)(x-4)^{-1} - x^2(x-4)^{-2} \\
&= \frac{2x}{x-4} - \frac{x^2}{(x-4)^2} \\
&= \frac{2x(x-4)}{(x-4)^2} - \frac{x^2}{(x-4)^2} \\
&= \frac{2x(x-4) - x^2}{(x-4)^2} \\
&= \frac{2x^2 - 8x - x^2}{(x-4)^2} \\
&= \frac{x^2 - 8x}{(x-4)^2} \\
&= \frac{x(x-8)}{(x-4)^2}
\end{aligned}
$$

$g'(x) = 0$ when $x = 0$ or $x = 8$.

$$\begin{aligned} g(0) &= \frac{0^2}{0-4} \\ &= 0 \end{aligned}$$

$$\begin{aligned} g(8) &= \frac{(8)^2}{8-4} \\ &= \frac{64}{4} \\ &= 16 \end{aligned}$$

(0, 0) and (8, 16) are two critical points of the function.

Since $(x - 4)^2 \geq 0$ for all x, you need worry about only the following intervals: $x < 0$, $0 < x < 8$, and $x > 8$. To check when the derivative is positive and negative, you need to check the value of the derivative for values of x in each of the intervals.

***x* < 0**

$$g'(x) = \frac{x(x-8)}{(x-4)^2}$$

$$\begin{aligned} g'(-1) &= \frac{(-1)((-1)-8)}{((-1)-4)^2} \\ &= \frac{9}{25} \end{aligned}$$

The derivative is positive and the function is increasing.

0 < *x* < 8

$g'(2) = -3$

The derivative is negative and the function is decreasing.

***x* > 8**

$g'(10) = \frac{5}{9}$

The derivative is positive and the function is increasing.

Since the function changes from increasing to decreasing at $x = 0$, you can assert that (0, 0) is a local maximum. Similarly, the function changes from decreasing to increasing at $x = 8$, so you can conclude that (8, 16) is a local minimum.

To find the second derivative, you have to differentiate the derivative. First, rewrite the derivative as a product and then use the product rule to differentiate.

$$\begin{aligned} g'(x) &= \frac{x(x-8)}{(x-4)^2} \\ &= x(x-8)(x-4)^{-2} \\ &= (x^2-8x)(x-4)^{-2} \end{aligned}$$

$$g''(x) = (2x-8)(x-4)^{-2} - 2(x^2-8x)(x-4)^{-3}$$

$$\begin{aligned} &\frac{(2x-8)}{(x-4)^2} - \frac{2(x^2-8x)}{(x-4)^3} \\ &= \frac{(2x-8)(x-4)}{(x-4)^3} - \frac{2(x^2-8x)}{(x-4)^3} \\ &= \frac{(2x-8)(x-4)-2(x^2-8x)}{(x-4)^3} \\ &= \frac{2x^2-8x-8x+32-2x^2+16x}{(x-4)^3} \\ &= \frac{32}{(x-4)^3} \end{aligned}$$

$g''(x) < 0$ when $x < 4$ and $g(x)$ is concave down.

$g''(x) > 0$ when $x > 4$ and $g(x)$ is concave up.

This confirms the assertion that (0, 0) is a local maximum and (8, 16) is a local minimum.

$g''(x)$ changes from positive to negative at $x = 4$, but there isn't an inflection point at $x = 4$. Why? $g(x)$ is not defined at $x = 4$, since $(x - 4)$ is in the denominator and dividing by zero is not allowed.

In the next example, you will describe properties of the graph of the function given information about its derivative.

Example

What do you know about the function $f(x)$, given that $f'(2) = 0$, $f'(5) = 0$, and $f'(x)$ is positive when $2 < x < 5$?

Solution

Since the derivative of the function is 0 at $x = 2$ and $x = 5$, you can conclude that the points $(2, f(2))$ and $(5, f(5))$ are critical points of the function. You can also assert that the function is increasing when x is between 2 and 5 because the derivative is positive.

Example

For the function $f(x) = 4x^3 - x^2 - 3$, determine

a) the intervals of increase or decrease

b) the location of any maximum or minimum points

c) the intervals of concavity up or down

d) the location of any points of inflection

Solution

a) Start by finding the first and second derivative:

$f'(x) = 12x^2 - 2x$

$f''(x) = 24x - 2$

$f'(x) = 0$

$12x^2 - 2x = 2x(6x - 1) = 0$

The zeros of the derivative occur when $x = 0$ or $6x - 1 = 0$.

$x = 0$ or $x = \frac{1}{6}$

The intervals you need to look at are $x < 0$, $0 < x < \frac{1}{6}$, and $x > \frac{1}{6}$.

$\boldsymbol{x < 0}$

Choose any value of x in this interval and substitute into the derivative:

$x = -1$

$f'(-1) = 12(-1)^2 - 2(-1)$

$= 12 + 2$

$= 14 > 0$ The derivative is positive and hence the function is increasing.

$\mathbf{0} < \boldsymbol{x} < \mathbf{\frac{1}{6}}$

Choose any value in this interval:

$x = \frac{1}{12}$

$f'(\frac{1}{12}) = 12(\frac{1}{12})^2 - 2(\frac{1}{12})$

$= (\frac{1}{12}) - \frac{1}{6}$

$= -(\frac{1}{12}) < 0$ The derivative is negative and hence the function is decreasing.

$\boldsymbol{x} > \mathbf{\frac{1}{6}}$

Choose any value in this interval:

$x = 1$

$f'(1) = 12(1)^2 - 2(1)$

$= 12 - 2$

$= 10 > 0$ The derivative is positive and hence the function is increasing.

You can summarize the information as shown in the following table:

	$x < 0$	$0 < x < \frac{1}{6}$	$x > \frac{1}{6}$
$f'(x)$	positive	negative	positive
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

b)

Local maximum at $x = 0$, $f(0) = -3$. $(0, -3)$ is a local maximum.

Local minimum at $x = \frac{1}{6}$, $f(\frac{1}{6}) = 4(\frac{1}{6})^3 - (\frac{1}{6})^2 - 3$

$= -\frac{650}{216}$. $(\frac{1}{6}, -\frac{650}{216})$ is a local minimum.

c) $f''(x) = 24x - 2$

$f''(x) = 0$ when $x = \frac{1}{12}$, $f''(x) < 0$ when $x < \frac{1}{12}$, and $f''(x) > 0$ when $x > \frac{1}{12}$. Therefore, the function is concave down when $x < \frac{1}{12}$ and concave up when $x > \frac{1}{12}$.

d) Since the second derivative goes from negative to positive at $x = \frac{1}{12}$, you can conclude that $(\frac{1}{12}, f(\frac{1}{12}))$ is a point of inflection of the function $f(x)$. $f(\frac{1}{12}) = -\frac{5192}{1728}$, so the point of inflection is $(\frac{1}{12}, -\frac{5192}{1728})$

Support Questions
(do not send in for evaluation)

13. For the function $f(x) = x^3 + 3x^2 - 24x + 1$, determine
    a) the intervals of increase or decrease
    b) the location of any maximum or minimum points
    c) the intervals of concavity up or down
    d) the location of any points of inflection
14. For the function $f(x) = x^4 - 2x^3 + 2x^2 - 2x + 4$, determine
    a) the intervals of increase or decrease
    b) the location of any critical points and classify the critical points as local maximum, local minimum, or neither
    c) the intervals of concavity up or down
    d) the location of any points of inflection

There are Suggested Answers to Support Questions at the end of this unit.

Determining the Equation of a Function

In the following example, you will start with some information about the graph of the function to determine its equation.

Example

A function has the following characteristics: $g''(-1) = 1, g'(2) = 0$ and $g(2) = 0$. Determine a possible equation for this function.

Solution

Since the second derivative is not zero, you can assume that the function is a polynomial of degree 2. Start with $g(x) = ax^2 + bx + c$ where a, b, and c are real numbers.

Use the usual differentiation rules to get the following. Note that a, b, and c are each treated like a constant number.

$g'(x) = 2ax + b$

$g''(x) = 2a$

Now substitute the given values: $g''(-1) = 1$, $g'(2) = 0$ and $g(2) = 0$.

$g'(x) = 2ax + b$

$g'(2) = 2a(2) + b$

$0 = 4a + b$ **[1]**

$g''(x) = 2a$

$g''(-1) = 2a$

$1 = 2a$

$a = \frac{1}{2}$

Substitute the value of $a = \frac{1}{2}$ into **[1]**:

$0 = 4\left(\frac{1}{2}\right) + b$

$b = -2$

$ax^2 + bx + c = 0$

$a(2)^2 + b(2) + c = 0$ **[2]**

Substitute $a = \frac{1}{2}$ and $b = -2$ into **[2]** and solve for c.

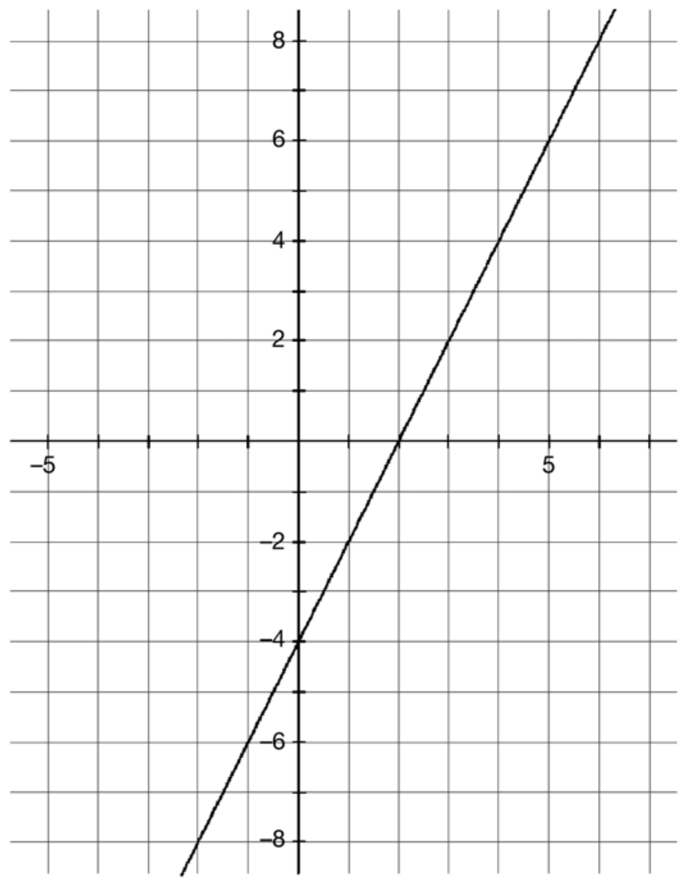
$4\left(\frac{1}{2}\right) - 4 + c = 0$

$c = 2$

A possible value of $g(x)$ is $\frac{1}{2}x^2 - 2x + 2$.

Example

The following graph is the graph of $f'(x)$. If $f(-1) = 2$, find the equation of $f(x)$.



Solution

From the graph, observe that $f'(x)$ is a linear function, which implies that the degree of $f'(x)$ is 1. Therefore, $f(x)$ is a quadratic function since $f(x)$ has one degree more than its derivative.

$f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$

In the graph, observe that $f'(2) = 0$ and $f'(0) = -4$. This gives you two equations with two unknowns:

$f'(2) = 2a(2) + b$

$0 = 4a + b$

$b = -4a$ **[1]**

$f'(0) = 2a(0) + b$

$-4 = b$ **[2]**

Substitute $b = -4$ into **[1]**:

$b = -4a$

$-4 = -4a$

$a = 1$

Substitute a and b into $f(x)$, $f(x) = x^2 - 4x + c$

How can you find the value of c? You know that $f(-1) = 2$

$(-1)^2 - 4(-1) + c = 2$

$5 + c = 2$

$c = -3$

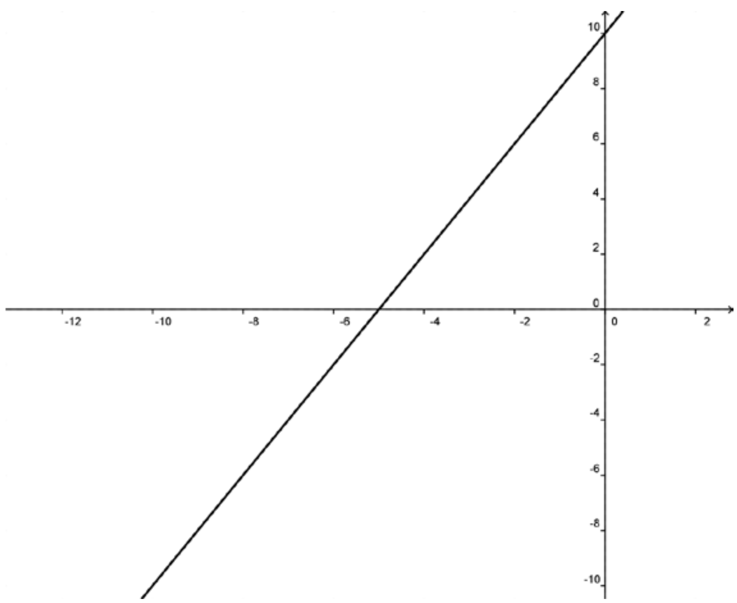
Therefore, $f(x) = x^2 - 4x - 3$.

Support Questions

(do not send in for evaluation)

15. A function has the following characteristics: $g''(2) = 1$, $g'(2) = 3$ and $g(2) = 0$. Determine a possible equation for this function.

16. The following is the graph of $h'(x)$. If $h(1) = 3$, find $h(x)$:



Conclusion

In Lesson 10, you will see how the techniques you have learned can be combined to help you sketch reasonable graphs of polynomial functions given the equation of the function. You do this by finding the intervals on which the function is increasing or decreasing, the critical points, inflection points, and other important features that you will study in the lesson.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(21 marks)

22. For the function $f(x) = x^3 - x^2 + 4x - 3$, determine:

    a) the intervals of increase or decrease **(2 marks)**

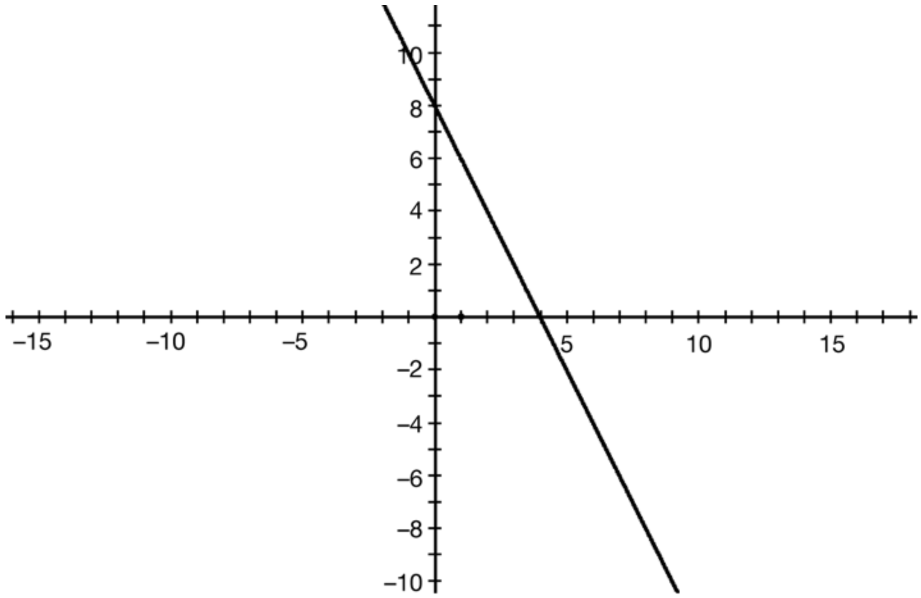
    b) the location of any maximum or minimum points **(1 mark)**

    c) the intervals of concavity up or down **(2 marks)**

    d) the location of any points of inflection **(1 mark)**

23. A quadratic function has the following characteristics: $g''(1) = 6$, $g'(1) = 3$, and $g(1) = 1$. Determine a possible equation for this function. **(8 marks)**

24. The following is the graph of $f'(x)$. If $f(2) = -1$, find $f(x)$. **(7 marks)**



Now go on to Lesson 10. Do not submit your coursework to ILC until you have completed Unit 2 (Lessons 6 to 10).

