MCV4U-A

8

The Second Derivative: A Graphical Look

Introduction

In Lesson 3, you learned how the instantaneous rate of change can give you information about the behaviour of a function. In this lesson, you will expand your knowledge of the derivative function and be introduced to the idea of a higher-order derivative. In particular, you will learn about the second derivative of a function and the information it gives you about the function itself.

Estimated Hours for Completing This Lesson	
Exploring the Relationship Between a Function and Its Derivatives	0.5
Higher-Order Derivatives	1.5
Concavity of a Function	2
Key Questions	1

For this lesson, there is an interactive tutorial on your course page. You may find it helpful during, or at the end of, this lesson to work through the tutorial called “First and Second Derivatives.”

What You Will Learn

After completing this lesson, you will be able to

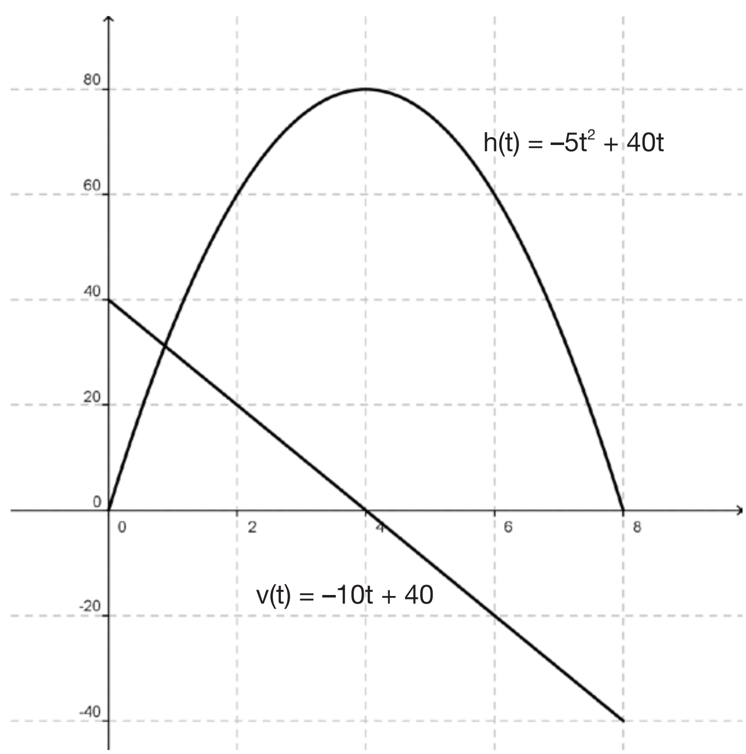
- make connections between the graph of the derivative of a function and the function itself
- draw a graph of the derivative given the graph of the function
- determine higher-order derivatives
- make connections between the graph of the second derivative of a function and the function itself

Exploring the Relationship Between a Function and Its Derivatives

Start with a familiar example involving a projectile thrown in the air. Suppose the following function represents the height in metres of a projectile at t seconds after its launch: $h(t) = -5t^2 + 40t$. The derivative of $h(t)$, $h'(t)$, gives you the instantaneous rate of change of the height with respect to time. What is another term for the rate of change of a distance? It is also called the velocity ($v(t)$) of the projectile:

$v(t) = h'(t) = -10t + 40$

Graph $h(t)$ and $v(t)$ on the same graph. Note that restrictions are imposed on the domain. The t variable is greater than or equal to zero since the time cannot be negative, and since the projectile is launched from the ground the height cannot be negative.



Recall from Lesson 3 that a positive instantaneous rate of change on an interval tells you that the function is increasing, while a negative instantaneous rate of change tells you that the function is decreasing. Look at the graph. What do you observe? The height of the projectile is increasing on the interval $0 < t < 4$ and decreasing on the interval $4 < t < 8$.

Look at the graph of the derivative (the line). What do you see? The derivative is greater than zero (above the x-axis) when $0 < t < 4$ and less than zero (below the x-axis) when $4 < t < 8$.

The velocity is positive when the projectile is going up and negative when it is going down. What happens to the projectile at 4 seconds? The projectile reaches its maximum height and starts falling down. There is an instant when the velocity is zero, the time when the projectile is changing direction.

Calculate the derivative of the velocity function. This is the acceleration function of the projectile with respect to t. The letter a is used for acceleration:

$a(t) = v'(t) = -10 \text{ m/s}^2$

The acceleration is a negative constant number, which makes sense since the velocity is continuously decreasing. Remember that the positive direction is upward, so even when the projectile is falling faster toward the ground after reaching its maximum height, the velocity is still decreasing.

Higher-Order Derivatives

The second derivative of a function is the derivative of the derivative of $f(x)$. The following notations are used: $f''(x)$, $\dfrac{d^2y}{dx^2}$, or y''. Similar notation is used for the third-order derivative, which is the derivative of the second derivative: $f'''(x)$, $\dfrac{d^3y}{dx^3}$, or y'''.

Examples

Find the first and second derivatives of the following functions:

a) $f(x) = x^3 - 2x + 3$

b) $y = \dfrac{1}{x^2 - 2}$

c) $g(x) = \dfrac{\sin(x)}{x - 1}$

Solutions

a) $f(x) = x^3 - 2x + 3$

$f'(x) = 3x^2 - 2$

$f''(x) = 6x$

b) $y = \dfrac{1}{x^2 - 2}$

Rewrite the question as $y = (x^2 - 2)^{-1}$ and use the chain rule to find the derivative:

$y' = -(x^2)'(x^2 - 2)^{-2}$

$y' = -2x(x^2 - 2)^{-2}$

To find the second derivative, use the product rule to differentiate y':

$y'' = (-2x)'(x^2 - 2)^{-2} + (-2x)((x^2 - 2)^{-2})'$

$y'' = -2(x^2 - 2)^{-2} + (-2x)(2x)(-2)(x^2 - 2)^{-3}$

$y'' = -2(x^2 - 2)^{-2} + 8x^2(x^2 - 2)^{-3}$

It is acceptable to leave the answer like this or to rewrite y'':

$$y'' = \frac{-2}{(x^2 - 2)^2} + \frac{8x^2}{(x^2 - 2)^3}$$

c) $g(x) = \dfrac{\sin(x)}{x-1}$

Rewrite the question as $g(x) = \sin(x)(x - 1)^{-1}$ and use the product rule to find the derivative.

$$\begin{aligned} g'(x) &= \sin(x)((x-1)^{-1})' + (\sin(x))'(x-1)^{-1} \\ &= \sin(x)(-1)(x-1)^{-2} + \cos(x)(x-1)^{-1} \\ &= -\sin(x)(x-1)^{-2} + \cos(x)(x-1)^{-1} \end{aligned}$$

Now differentiate $g'(x)$.

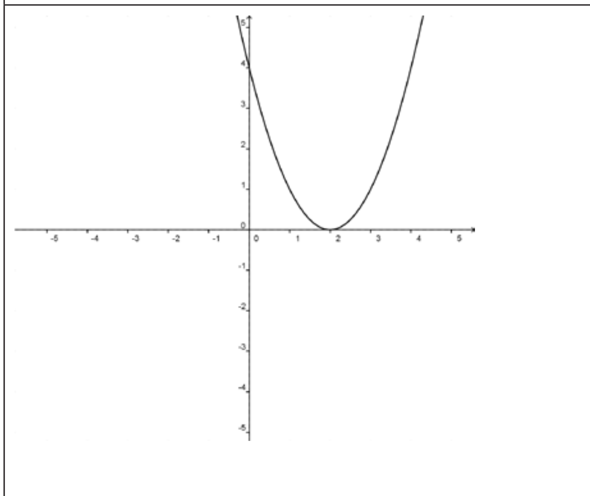
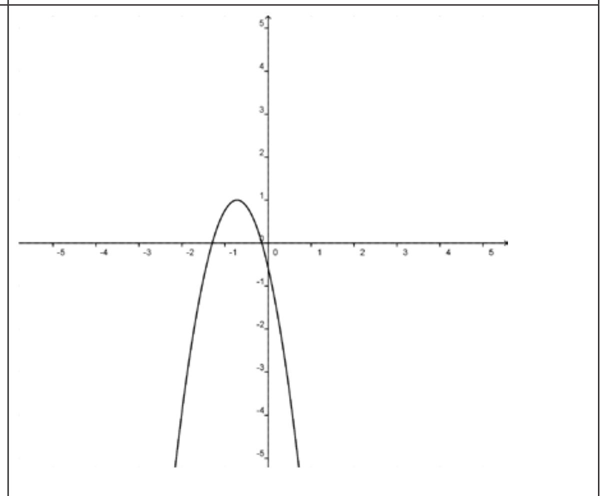
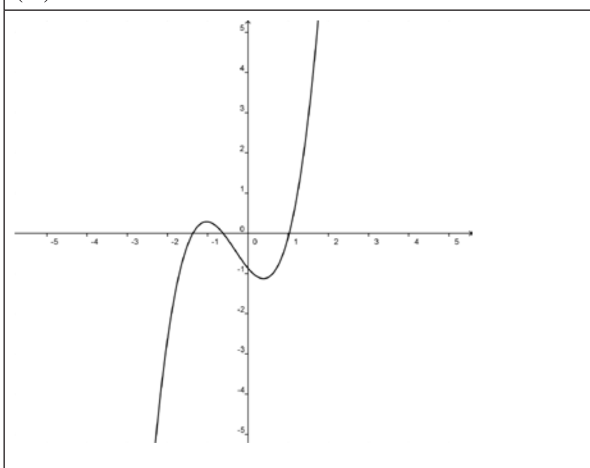
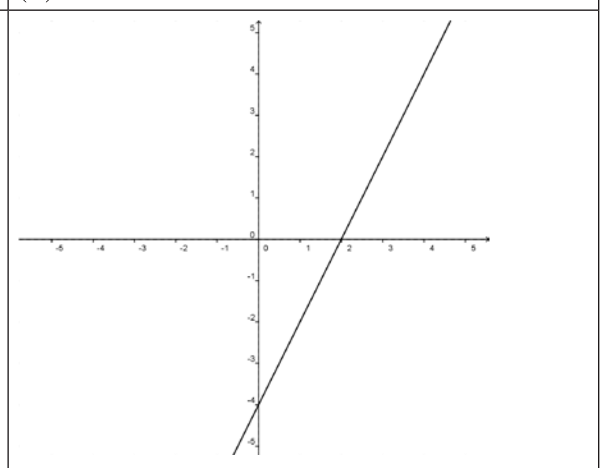
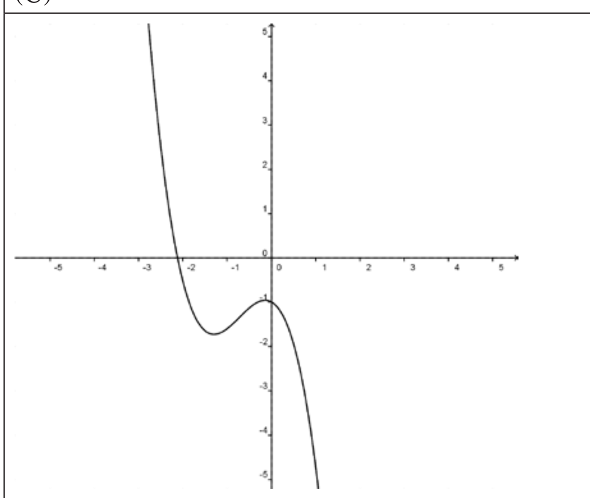
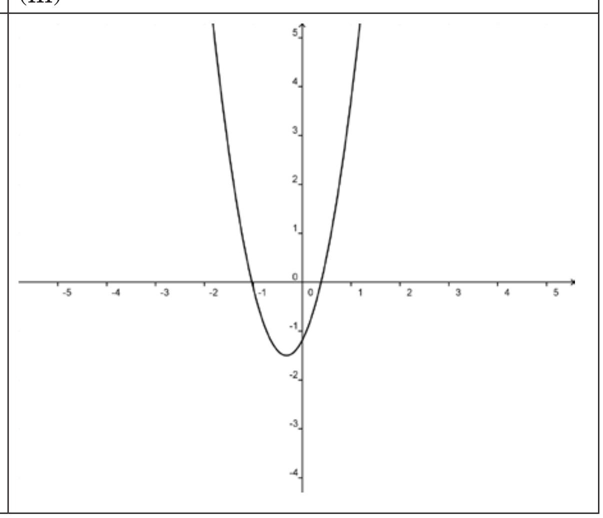
$$g''(x) = -\sin(x)(-2)(x-1)^{-3} - \cos(x)(x-1)^{-2} + (-1)\cos(x)(x-1)^{-2} - \sin(x)(x-1)^{-1}$$

$$g''(x) = \frac{2\sin(x)}{(x-1)^3} - \frac{2\cos(x)}{(x-1)^2} - \frac{\sin(x)}{(x-1)}$$

In the next example, you will match the graph of the function with the graph of its derivative by looking at the intervals where the function is increasing or decreasing.

Examples

Match the graph of each function (A), (B), or (C) with the graph of its derivative (i), (ii), or (iii).

(A)		(i)
		
(B)		(ii)
		
(C)		(iii)
		

Solutions

(A): the derivative is (ii)

The function in (A) is decreasing when $x < 2$ and increasing when $x > 2$. The function in (ii) is less than zero when $x < 2$ and larger than zero when $x > 2$.

(B): the derivative is (iii)

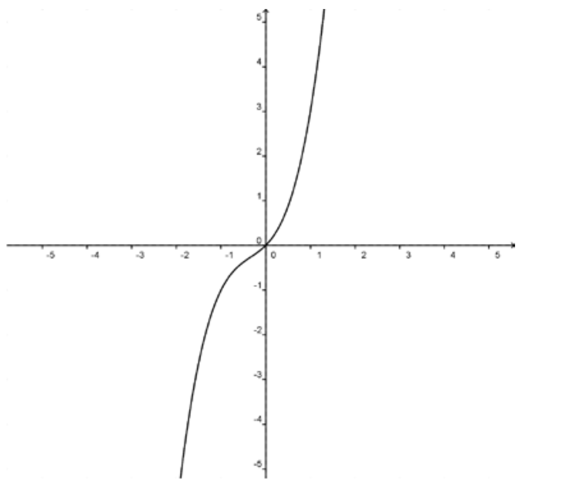
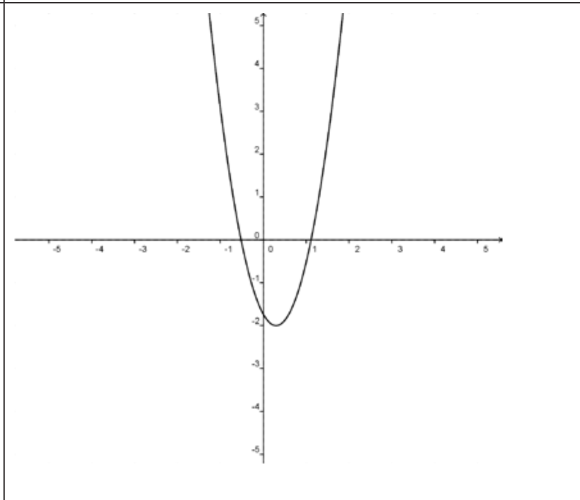
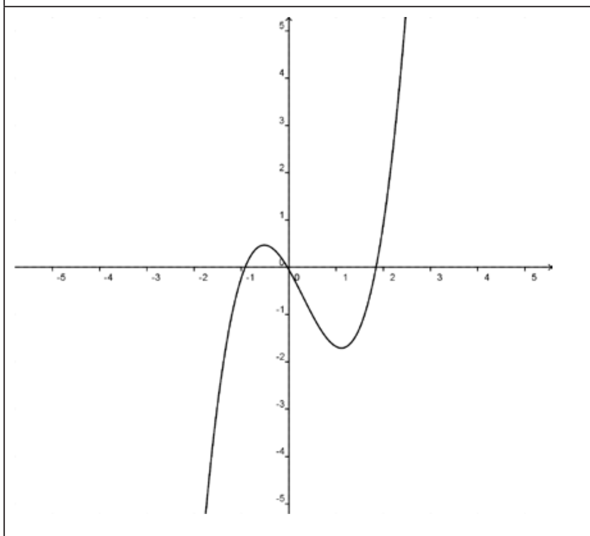
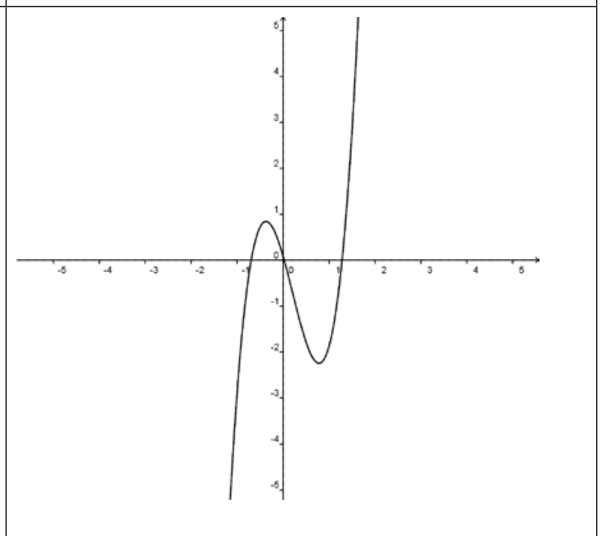
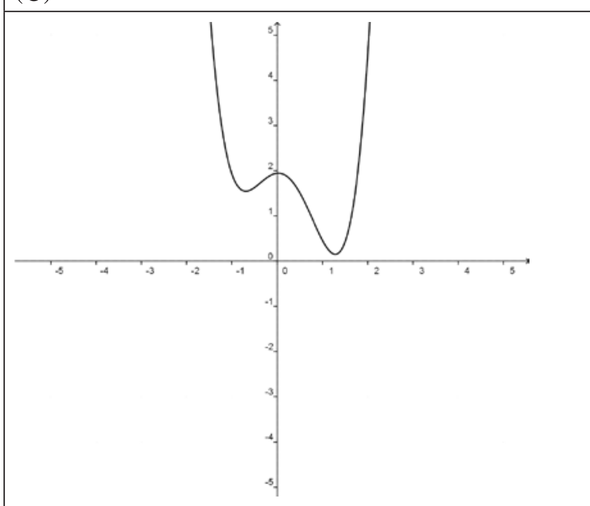
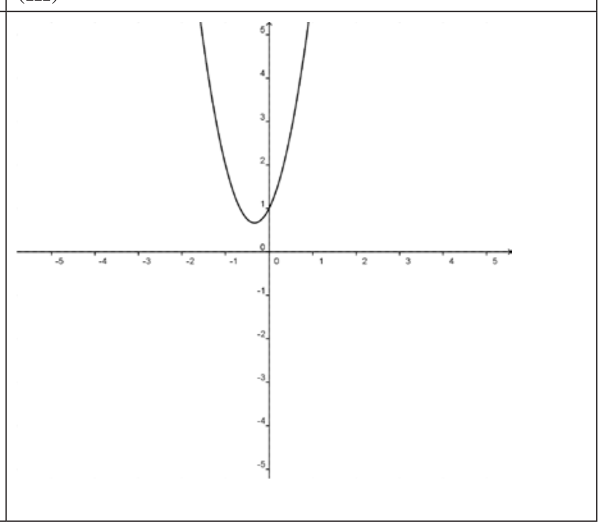
The function in (B) is increasing when $x < -1$, decreasing when $-1 < x < 0.3$, and increasing again when $x > 0.3$. The function in (iii) is positive when $x < -1$ and $x > 0.3$, and negative when $-1 < x < 0.3$.

(C): the derivative is (i)

The function in (C) is decreasing when $x < -1.2$ and $x > -0.1$, and increasing when $-1.2 < x < -0.1$. The function in (i) is negative when $x < -1.2$ and $x > -0.1$, and positive when $-1.2 < x < -0.1$.

Support Question
(do not send in for evaluation)

9. Match the graph of each function (A), (B), or (C) with the graph of its derivative (i), (ii), or (iii).

(A)		(i)
		
(B)		(ii)
		
(C)		(iii)
		

There are Suggested Answers to Support Questions at the end of this unit.

Concavity of a Function

You are now going to study the connection between the second derivative of a function and the function itself. Look at the graph of the following function and the graph of its second derivative.

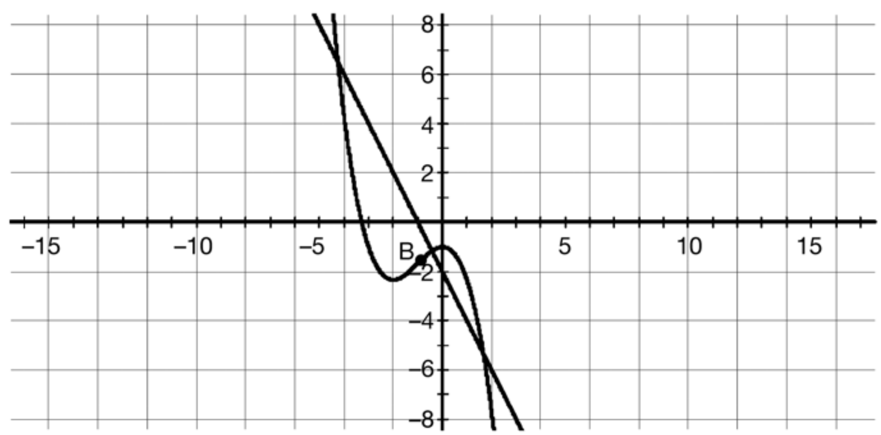
First, find the first and second derivative of $f(x) = -\frac{1}{3}x^3 - x^2 - 1$:

$f'(x) = 3(-\frac{1}{3})x^2 - 2x$

$f'(x) = -x^2 - 2x$

$f''(x) = -2x - 2$

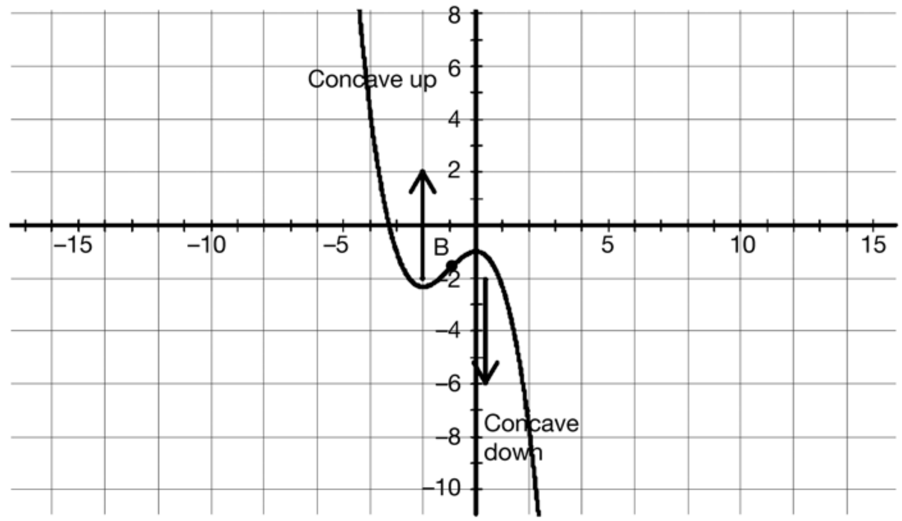
The following figure shows the graph of $f(x) = -\frac{1}{3}x^3 - x^2 - 1$ and its second derivative $f''(x) = -2x - 2$:



The second derivative is zero when $x = -1$. You can assert this by solving: $-2x - 2 = 0 \rightarrow x = -1$. You can also observe that $f''(x) > 0$ when $x < -1$ and $f''(x)$ is < 0 when $x > -1$.

Look at the graph of the function itself. What is happening when $x = -1$?

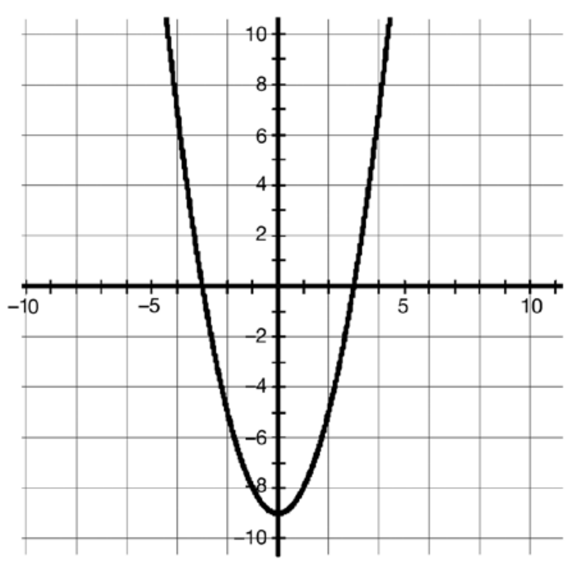
The function opens upward (concave up) when $x < -1$ and downward (concave down) when $x > -1$. It is concave up when the second derivative is positive and concave down when the second derivative is negative.



Point B, where the concavity changes, is called a point of inflection.

Support Question
(do not send in for evaluation)

10. The following graph shows $f'(x)$.
    a) When is $f(x)$ increasing and decreasing?
    b) Draw a sketch of $f(x)$.
    c) Where is the inflection point of $f(x)$?



Important Facts

$f(x)$ is increasing when $f'(x) > 0$

$f(x)$ is decreasing when $f'(x) < 0$

If $f'(a) = 0$ or $f'(x)$ is not defined at $x = a$, then the point $(a, f(a))$ is a critical point of f.

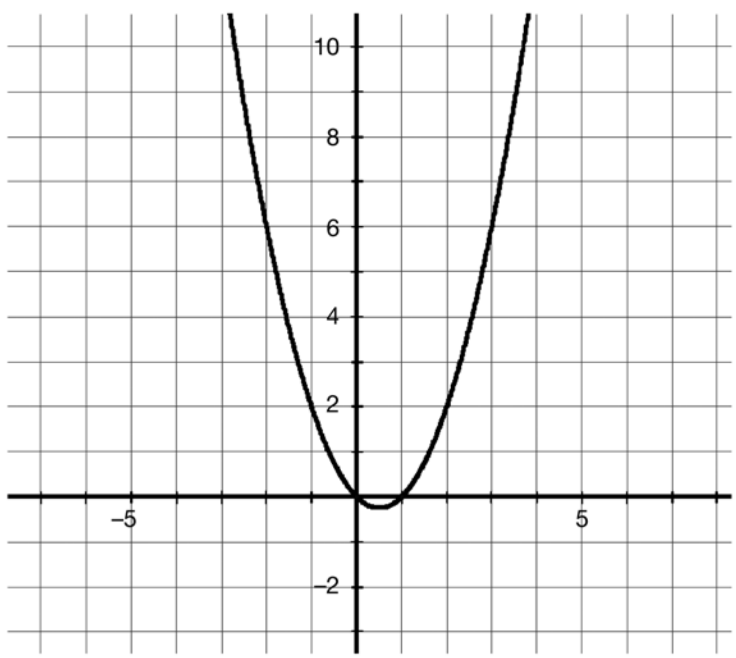
$f(x)$ is concave up when $f''(x) > 0$

$f(x)$ is concave down when $f''(x) < 0$

If $f''(x)$ changes sign at $x = c$ (from negative to positive or from positive to negative) then the point $(c, f(c))$ is an inflection point of f.

Example

Based on the following graph of $f'(x)$, when is $f(x)$ increasing and decreasing? Draw a sketch of $f(x)$. Where is the inflection point of $f(x)$?



Solution

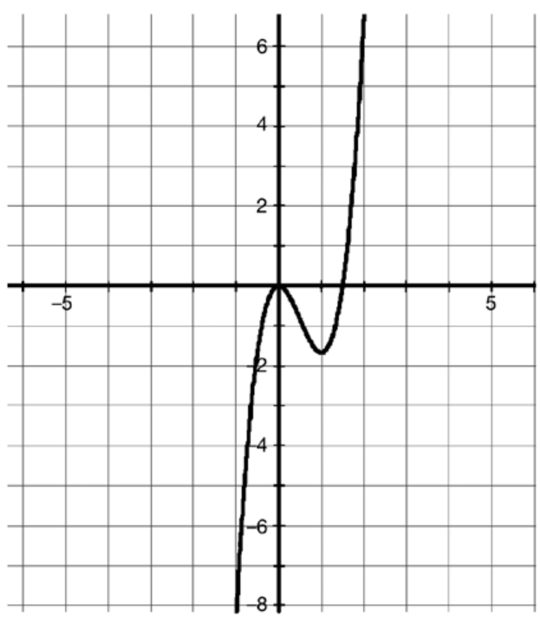
$f'(x) > 0$ when $x < 0$ and $x > 1$, and $f'(x) < 0$ when $0 < x < 1$. The derivative is zero at $x = 0$ and $x = 1$.

Notice that the slope of the tangent to the curve at $x = \frac{1}{2}$ is zero. You can conclude that the derivative of the derivative function (that is, the second derivative) is zero when $x = \frac{1}{2}$. You can also see that $f'(x)$ is decreasing when $x < \frac{1}{2}$, so $f''(x)$ is negative there and $f'(x)$ is increasing when $x > \frac{1}{2}$, so $f''(x)$ is positive there. Therefore, $f(x)$ has an inflection point at $x = \frac{1}{2}$.

The information for this example is summarized in the table:

	$x < 0$	$0 < x < 1$	$x > 1$
$f'(x)$	positive	negative	positive
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

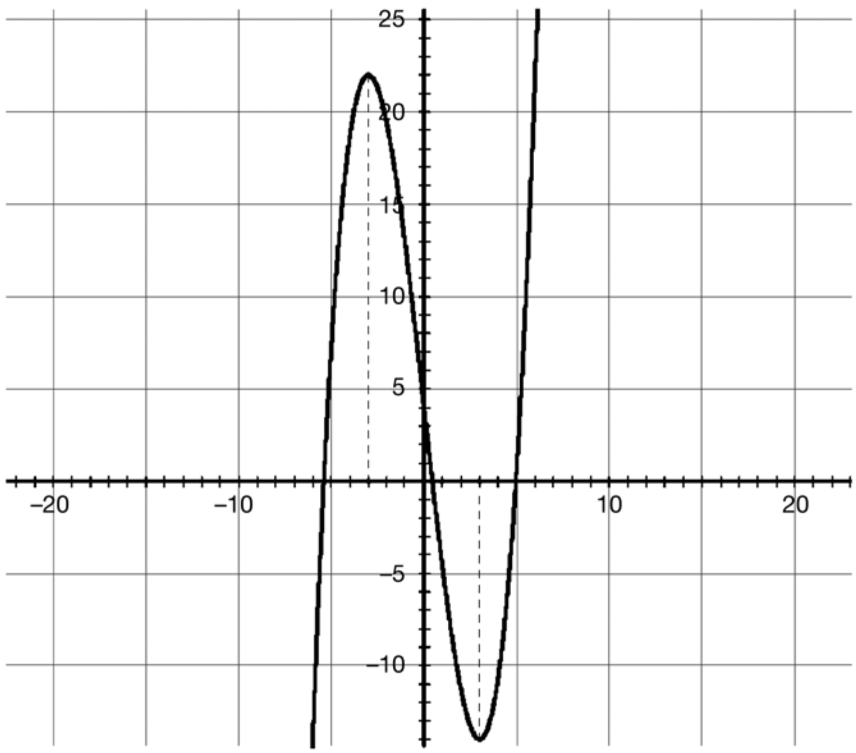
The following graph shows a function that satisfies the conditions summarized in the table:



You can also use information from the graph of a first derivative to draw a graph of the second derivative.

Example

Given the graph of $f(x)$, a polynomial of degree 3, draw a graph of the derivative of $f(x)$ and its second derivative.



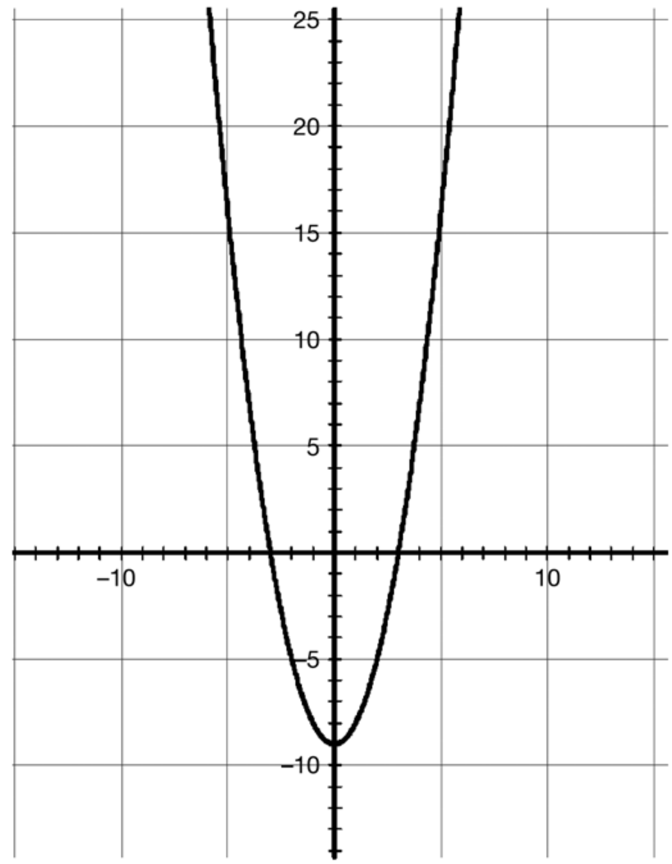
Solution

You know that $f(x)$ is a polynomial of degree 3, so its derivative is a quadratic, and the graph of a quadratic function is a parabola.

The function is increasing for $x < -3$ and $x > 3$, and decreasing when $-3 < x < 3$. You can conclude that the derivative is positive when $x < -3$ and $x > 3$, and it is negative when $-3 < x < 3$.

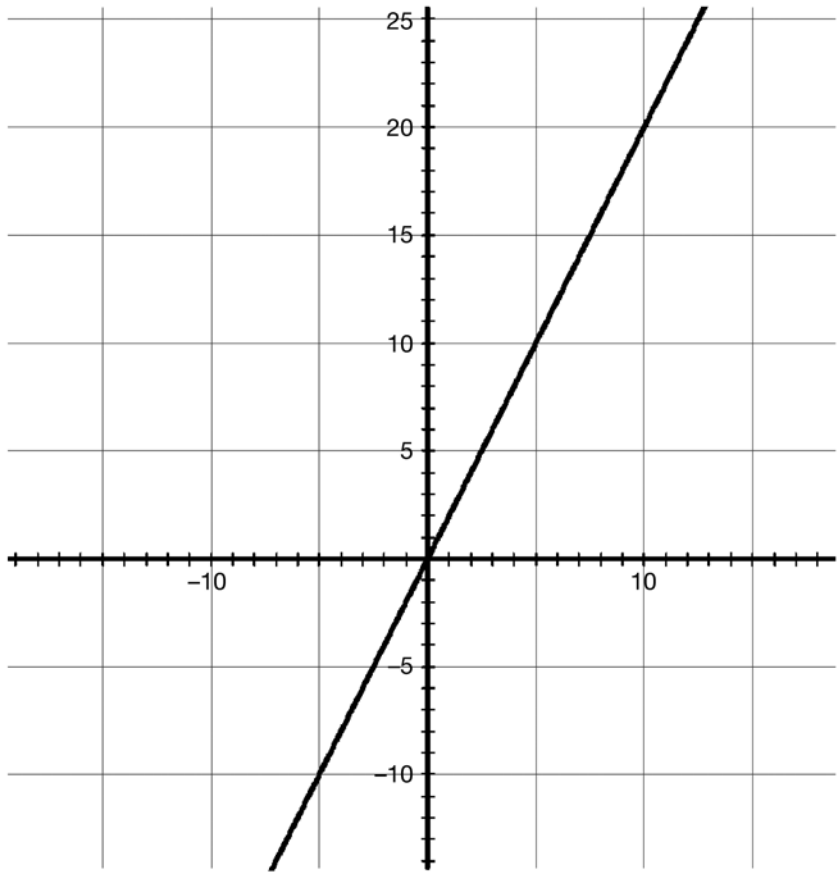
The function has a critical point at $x = -3$ and $x = 3$ since the slope of the tangent to the curve is zero at both points (that is, the derivative is zero at $x = -3$ and $x = 3$).

The following graph represents the first derivative:



You can also observe that the function is concave down when $x < 0$ and concave up when $x > 0$ and that there is an inflection point at $x = 0$.

Since $f(x)$ is concave up when $x > 0$ and concave down when $x < 0$, the following graph represents the second derivative: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20
15
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5
–10
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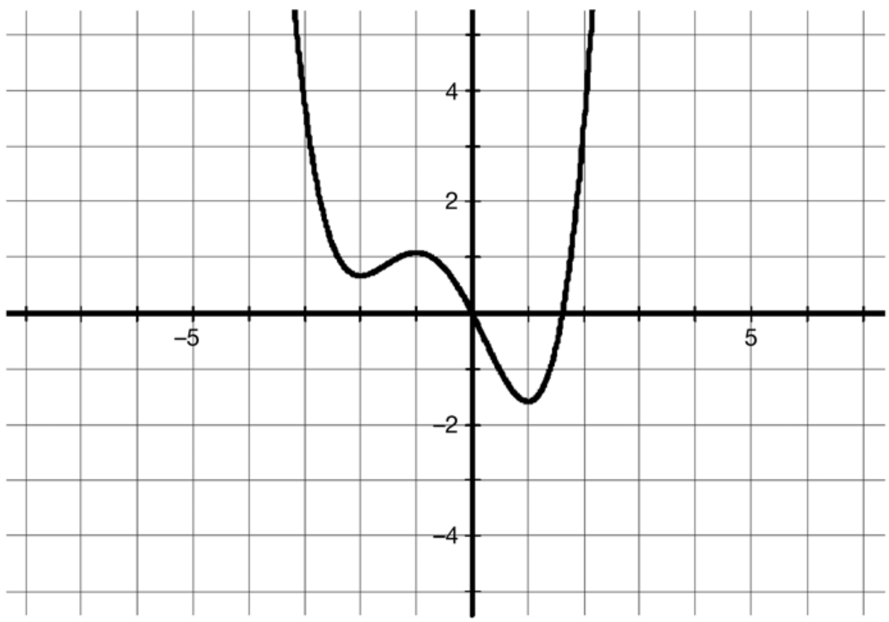
Support Questions
(do not send in for evaluation)

11. Given the graph of $f(x)$, a polynomial of degree 4, draw a graph of $f'(x)$ and $f''(x)$.



12. Find the first and second derivatives of the following functions:

 a) $y = 4x^6 - 4x^2 - 5x$

 b) $f(x) = \sqrt{x^2 - 3}$

 c) $g(x) = \dfrac{x^2 - 1}{x}$

Conclusion

In this lesson, you learned how the second derivative of a function helps you learn about the behaviour of the function itself. You learned that if the derivative is positive in an interval, then the function is increasing and a negative derivative means that the function is decreasing. You also learned about the second derivative of a function and how it gives you information about the concavity of a function. In the next lesson, you will study the behaviour of a function starting with the algebraic representation (equation) of the function.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(23 marks)

19. Determine the first and second derivatives of the following functions:

    a) $y = (3x + 2)^2$ **(2 marks)**

    b) $f(x) = 5x^2 - 2x$ **(2 marks)**

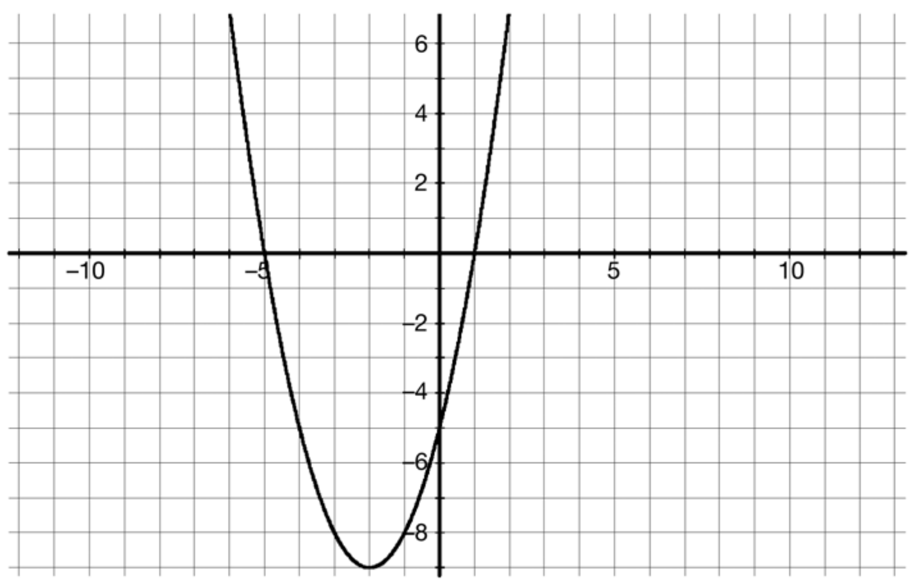
    c) $g(x) = \dfrac{2x - 3}{x + 4}$ **(3 marks)**

20. Given the graph of $g'(x)$, use graph paper to sketch $g''(x)$ and a possible graph of $g(x)$. Identify the following:

    a) The intervals where $g(x)$ is increasing and decreasing **(2 marks)**

    b) The local maximum and minimum points of $g(x)$ **(1 mark)**

    c) The intervals where $g(x)$ is concave up and concave down **(5 marks)**



21. Given the following graph of $h(x)$, identify:

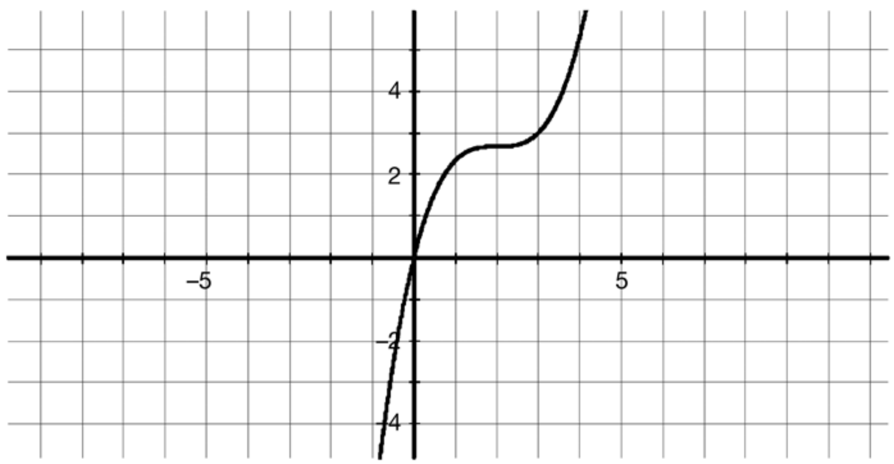
    a) The intervals where $h(x)$ is increasing and decreasing **(1 mark)**

    b) The local maximum and minimum points of $h(x)$ **(2 marks)**

    c) The intervals where $h(x)$ is concave up and concave down **(1 mark)**

    d) The inflection points of $h(x)$ **(1 mark)**

    e) Sketch the graphs of $h'(x)$ and $h''(x)$. **(3 marks: 1 mark for the first graph, 2 marks for the second graph)**



Now go on to Lesson 9. Do not submit your coursework to ILC until you have completed Unit 2 (Lessons 6 to 10).