MCV4U-A

7

The Derivatives of Sinusoidal and Rational Functions

Introduction

In this lesson, you explore the graphs of sinusoidal functions and functions of the form x^n, where n is a rational number. You examine the slope of the tangent lines to these graphs and learn the formulas for differentiating sinusoidal functions. You also see how the various differentiation rules you have learned can be combined to find the derivatives of more complicated functions.

Estimated Hours for Completing This Lesson	
The Derivatives of Sine and Cosine Functions	1
Exploring the Graph of x^n Where n Is a Fraction	1
Derivatives of More Complicated Functions	2
Key Questions	1

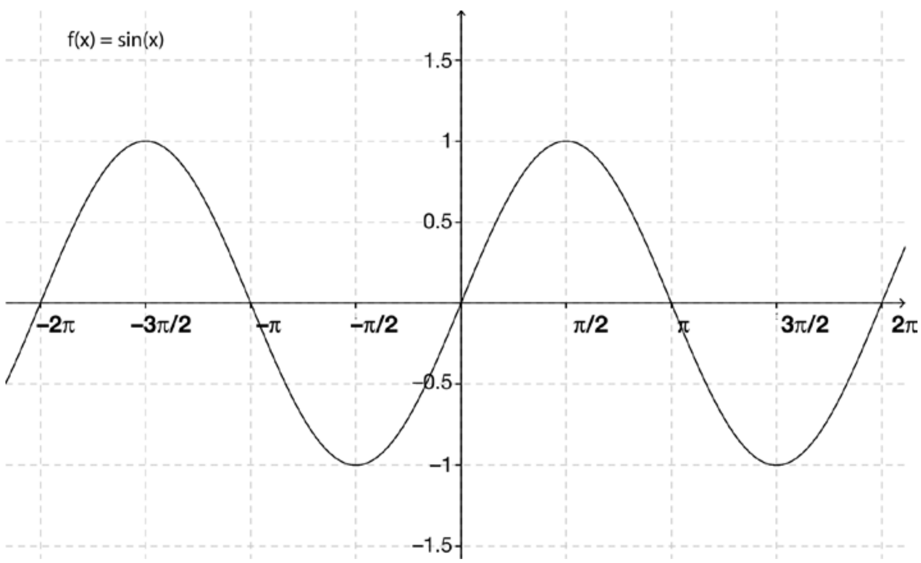
What You Will Learn

After completing this lesson, you will be able to

- find the derivatives of sinusoidal functions
- find the derivatives of functions built out of sinusoidal functions, exponential functions, polynomial functions, and functions of the form x^n, where n is a fraction, by using rules previously learned, such as the product rule, the power rule, and the chain rule

The Derivatives of Sine and Cosine Functions

Review the graphs of the familiar trigonometric functions, paying special attention to the slope of the tangent line at various points. You've previously seen the graph of the sine function, and are familiar with some of the special points on the graph. Here, following the usual convention in calculus, you are using radians rather than degrees. Recall that π radians is the same as 180 degrees (so $\frac{\pi}{2}$ radians is the same as 90 degrees, $\frac{\pi}{3}$ radians is the same as 60 degrees, $\frac{3\pi}{2}$ radians is the same as 270 degrees, and so on).



Some of the special values of the sine function are summarized in the following chart:

x	$\sin x$
$-\frac{3\pi}{2}$	1
$-\pi$	0
$-\frac{\pi}{2}$	–1
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	–1
2π	0
$\frac{5\pi}{2}$	1

You may also recall other special values of the sine function, such as $\sin\frac{\pi}{6} = \sin 30° = \frac{1}{2}$, $\sin\frac{\pi}{4} = \sin 45° = \frac{1}{\sqrt{2}}$, and $\sin\frac{\pi}{3} = \sin 60° = \frac{\sqrt{3}}{2}$. In any case, the focus now is on the question of the derivative of the sine function, so the main interest is the slope of the tangent line to the sine graph at various points.

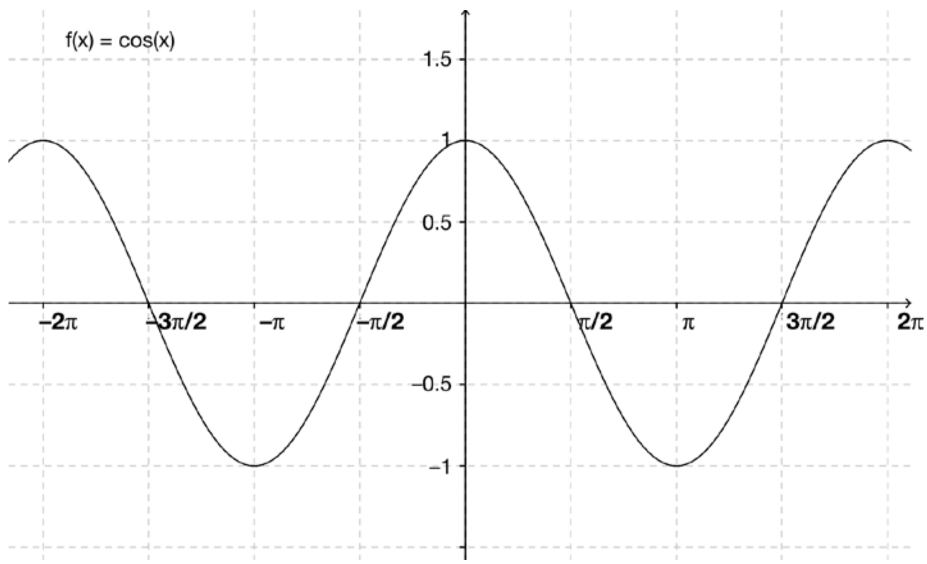
Notice the following from the graph you just looked at:

- At the points where $\sin(x) = +1$ or -1 (the maximum and minimum points on the graph), the curve has a horizontal tangent line, so the slope is zero at those points.
- The graph appears to have its steepest increase at $x = 0$, $x = 2\pi$, and so on (as well as $x = -2\pi, -4\pi, \ldots$). The slope at these points appears to be approximately 1.
- The graph appears to have its steepest decrease at $x = \pi$, $x = 3\pi$, and so on (as well as $x = -\pi, -3\pi, \ldots$). The slope at these points appears to be approximately –1.

Open “Lesson 7 Activity 1” on your course page at *ilc.org*, with which you can explore the graph of the sine function a bit more thoroughly. Attached to point A is the tangent line to the curve at A. The tangent line will move accordingly as point A is moved. As you move A, you can observe the coordinates of A, as well as the value of the slope of the tangent line. If you keep track of your observations in a table, you should see something like the following. (Due to rounding and the imprecision of moving points by hand, your numbers might not agree exactly with those in the table.)

Value of x	Slope of tangent line to $y = \sin x$
0	1
0.1	0.995
0.2	0.980
0.3	0.955
0.4	0.921
0.5	0.878
0.6	0.825
0.7	0.765
0.8	0.697
0.9	0.622
1.0	0.540
1.1	0.454
1.2	0.362

Perhaps nothing is immediately obvious just from looking at this table of numbers, but there is an extremely simple formula for the derivative of the sine function. Before it is stated here, review some features of the graph of the cosine function.



Here is a summary of some of the special values of the cosine function:

x	$\cos x$
-2π	1
$-\frac{3\pi}{2}$	0
$-\pi$	–1
$-\frac{\pi}{2}$	0
0	1
$\frac{\pi}{2}$	0
π	–1
$\frac{3\pi}{2}$	0
2π	1
$\frac{5\pi}{2}$	0

You may notice something if you compare the slope of the sine graph with the values of the cosine function. At each of the special points $\ldots, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$, the slope of the sine graph is equal to the value of the cosine function. In fact, this is true for all values of x.

To summarize:

Derivative of the sine function:

The derivative of the function $f(x) = \sin x$ is $f'(x) = \cos x$.

It's possible to prove this rule algebraically, but the proof is beyond what you can do in this course.

Similarly, you can study the derivative of the cosine function. Here's a table that summarizes the value of slope of the tangent to $\cos x$ at various points and the value of $-\sin x$.

x	Slope of tangent to $y = \cos x$	Value of $-\sin x$
$-\frac{3\pi}{2}$	–1	–1
$-\pi$	0	0
$-\frac{\pi}{2}$	1	1
0	0	0
$\frac{\pi}{2}$	–1	–1
π	0	0
$\frac{3\pi}{2}$	1	1
2π	0	0
$\frac{5\pi}{2}$	–1	–1

x	Slope of tangent to $y = \cos x$	Value of $-\sin x$
0	0	0
0.2	–0.199	–0.199
0.4	–0.389	–0.389
0.6	–0.565	–0.565
0.8	–0.717	–0.717
1.0	–0.841	–0.841
1.2	–0.932	–0.932
1.4	–0.985	–0.985
1.6	–0.999	–0.999

Derivative of the cosine function:

The derivative of the function $f(x) = \cos x$ is $f'(x) = -\sin x$.

Now that you know the derivatives of the sine and cosine function, you can combine that knowledge with your previous exposure to the chain rule and other functions that you have already learned how to differentiate.

Examples

Determine the derivative of the following functions:

a) $f(x) = \sin(e^x)$

b) $g(x) = (\cos x)^3$

Solutions

a) This is a situation similar to the power of a function application you learned in Lesson 5. The outer function is the sine function and the inner function is e^x. The derivative is $f'(x) = \cos(e^x)e^x$.

b) This is the power of a function application you learned in Lesson 5. The outer function is the cubing function and the inner function is $\cos x$. The derivative is $g'(x) = 3(\cos x)^2(-\sin x) = -3\cos^2 x \sin x$.

Support Question
(do not send in for evaluation)

5. Determine the derivative of each of the following functions:

a) $f(x) = \sin(x^2)$

b) $g(x) = e^{\cos x}$

There are Suggested Answers to Support Questions at the end of this unit.

Exploring the Graph of x^n Where n Is a Fraction

In Lesson 5, you were introduced to the power rule and learned that it is valid for any real-number exponent, including fractional exponents. Here, you will spend some time exploring the graphs of functions of the form x^n, where n is a fraction, to verify that the rule you've learned is consistent with the behaviour of the graph.

Consider the graphs of the function $f(x) = x^{\frac{1}{2}}$ and the function $g(x) = x^{\frac{4}{3}}$.

Open "Lesson 7 Activity 2" on your course page at *ilc.org*, which consists of the graph of the function $f(x) = x^{\frac{1}{2}}$, together with a point A on the graph that you can move around and the tangent line to the graph at that point.

From the previous discussion of the power rule for fractional exponents, you know that the derivative of $f(x) = x^{\frac{1}{2}}$ is $f'(x) = \left(\frac{1}{2}\right)x^{-\frac{1}{2}}$, which can also be written $f'(x) = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$.

You can verify that when you move the point A and keep track of the slope, you will get the same values as when you plug the appropriate value of x into the formula $f'(x) = \left(\frac{1}{2}\right)x^{-\frac{1}{2}}$.

Value of x	Slope of tangent to $y = x^{\frac{1}{2}}$	Value of $\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$
1	0.500	$\frac{1}{2} = 0.500$
2.5	0.316	0.316
4	0.250	$\frac{1}{4} = 0.250$
6.5	0.196	0.196
9	0.167	$\frac{1}{6} \approx 0.167$

What happens to the expression $f'(x) = \left(\frac{1}{2}\right)x^{-\frac{1}{2}}$ when x becomes very large (say, $x = 1000$ or 1 000 000)? What about when x is very close to zero (say, $x = 0.001$ or 0.000001)? Is this consistent with what you can see about the slope of the graph of $y = x^{\frac{1}{2}}$?

As x gets very large, the derivative approaches zero, which is consistent with the graph $f(x) = x^{\frac{1}{2}}$. The slope of the tangent to the curve approaches a horizontal line, which has slope zero.

For further illustration, open "Lesson 7 Activity 3" on your course page at *ilc.org*, which is essentially the same thing as the previous example, but for the function $g(x) = x^{\frac{4}{3}}$.

As with the previous example, when you move point A around and keep track of the slope at various points, you can verify that the slope appears to be consistent with the power rule you have previously learned, which states that the derivative of $g(x) = x^{\frac{4}{3}}$ is given by $g'(x) = \left(\frac{4}{3}\right)x^{\frac{1}{3}}$.

Value of x	Slope of tangent to $y = x^{\frac{4}{3}}$	Value of $\left(\frac{4}{3}\right)x^{\frac{1}{3}}$
0.73	1.20	1.20
1	1.33	1.33
2	1.68	1.68
3.5	2.02	2.02
4.5	2.20	2.20

What will happen to the expression $g'(x) = \left(\frac{4}{3}\right)x^{\frac{1}{3}}$ when x becomes very large (say, $x = 1000$ or 1 000 000)? What about when $x = 0$? Is this consistent with what you can see about the slope of the graph of $y = x^{\frac{4}{3}}$?

When x becomes very large, so does $g'(x) = \left(\frac{4}{3}\right)x^{\frac{1}{3}}$, which is consistent with the graph since the tangent approaches a vertical line when x is large. Similarly, at $x = 0$, the $\left(\frac{4}{3}\right)x^{\frac{1}{3}}$ approaches zero and on the graph the tangent is horizontal.

You may have noticed that the graph of $y = x^{\frac{1}{2}}$ contains no points with negative values of x, whereas the graph of $y = x^{\frac{4}{3}}$ is defined for all values of x. This is because $x^{\frac{1}{2}}$ is the same thing as the square root of x, which is undefined when x is negative. On the other hand, $x^{\frac{4}{3}}$ means the same as $\left(x^{\frac{1}{3}}\right)^4$, which is the fourth power of the cube root of x. The cube root (or fifth root, or seventh root) is defined for all values of x, positive or negative. Since you are also raising to the fourth power, the end result is positive regardless of whether x is positive or negative.

Derivatives of More Complicated Functions

In the last part of this lesson, you will see how the rules you learned for differentiating functions can be used together to find the derivatives of a wide variety of complex-looking functions that are built out of simpler functions.

One type of function that is covered here is a "rational function." A rational function is any function consisting of a polynomial divided by another polynomial, such as $\dfrac{x^2 + 5x - 6}{x^3 - 3x + 7}$.

Examples

Determine the derivative of the following functions:

a) $y = \dfrac{x^2 + 5x - 6}{x^3 - 3x^2 + 7}$

b) $y = \left(5x^2 + 3\right)^{\frac{2}{3}}$

Solutions

a) You can find the derivative by first rewriting the function as $y = (x^2 + 5x - 6)(x^3 - 3x^2 + 7)^{-1}$

Now apply the product rule:

$y' = (x^2 + 5x - 6)'(x^3 - 3x^2 + 7)^{-1} + (x^2 + 5x - 6)((x^3 - 3x^2 + 7)^{-1})'$

Notice that the last step involves finding the derivative of something of the form u^{-1}, where u is a function. By the chain rule, you know that the derivative of u^{-1} is $(-1)u^{-2} \cdot u'$. You can conclude that the answer to the question is
$y' = (2x + 5)(x^3 - 3x^2 + 7)^{-1} +$
$(x^2 + 5x - 6)(-1)(x^3 - 3x^2 + 7)^{-2}(3x^2 - 6x)$.

Typically, you leave the final answer in this form (you could expand it, but that would be more work and wouldn't make the expression any simpler).

b) A form of the chain rule is used here as well. This time, you must differentiate a function of the form $u^{\frac{2}{3}}$, where u is a function. By the chain rule, the derivative is $\left(\frac{2}{3}\right)u^{-\frac{1}{3}} \cdot u'$. You can conclude that the answer to the question is $y' = \frac{2}{3}(5x^2 + 3)^{-\frac{1}{3}} \cdot 10x$, which can also be written $y' = \frac{20x}{3(5x^2 + 3)^{\frac{1}{3}}}$.

Support Question
(do not send in for evaluation)

6. Determine the derivative of the following functions:

a) $y = \frac{1}{x^3 - 3x^2 + 6}$

b) $y = \frac{x - 2}{x + 4}$

c) $y = (3x^2 - 4)^{\frac{1}{2}}$

Derivatives of Complex Functions

You can differentiate complex functions that contain an exponential or sinusoidal function.

Examples

Find the derivative of each of the following:

a) $f(x) = \dfrac{x}{\cos x^3}$

b) $g(x) = \dfrac{e^x}{\sqrt{3x^2 + 4}}$

Solutions

a) As in the previous set of examples, since this function is a quotient you can find the derivative by first rewriting the function in the form $f(x) = x(\cos x^3)^{-1}$. Then you apply the product rule to conclude that $f'(x) = (x)'(\cos x^3)^{-1} + x((\cos x^3)^{-1})'$. After one application of the chain rule with the "outer function" u^{-1}, you have
$f'(x) = 1 \cdot (\cos x^3)^{-1} + x(-1)(\cos x^3)^{-2} \cdot (\cos x^3)'$.

You're not quite done: you still have to find the derivative of $\cos x^3$. Do this with one more application of the chain rule. This time the outer function is the cosine function and the inner function is x^3. This gives you
$f'(x) = 1 \cdot (\cos x^3)^{-1} + x(-1)(\cos x^3)^{-2} \cdot (-\sin x^3) \cdot 3x^2$.

Now you're done. As in an earlier example, this doesn't simplify much further and it's entirely appropriate to leave it as written. Your answer can perhaps be rearranged a little bit into something like this:

$$f'(x) = \frac{1}{\cos x^3} + \frac{3x^3 \sin x^3}{(\cos x^3)^2}$$

b) Again, the function you're trying to differentiate can be rewritten as a product: $g(x) = e^x \cdot (3x^2 + 4)^{-\frac{1}{2}}$.

By the product rule, the derivative is

$$g'(x) = (e^x)'\left(3x^2 + 4\right)^{-\frac{1}{2}} + e^x\left(\left(3x^2 + 4\right)^{-\frac{1}{2}}\right)'.$$

By the chain rule, the derivative of something of the form $u^{-\frac{1}{2}}$ is $\left(-\frac{1}{2}\right)u^{-\frac{3}{2}} \cdot u'$. Recalling that the derivative of e^x is simply e^x, you can conclude that the answer to the question is $g'(x) = e^x\left(3x^2+4\right)^{-\frac{1}{2}} + e^x\left(-\frac{1}{2}\right)\left(3x^2+4\right)^{-\frac{3}{2}}(6x)$.

Again, it is appropriate not to try to simplify, but your answer can also be rearranged slightly:

$$g'(x) = \frac{e^x}{(3x^2+4)^{\frac{1}{2}}} - \frac{3xe^x}{(3x^2+4)^{\frac{3}{2}}}.$$

Support Question
(do not send in for evaluation)

7. Find the derivative of each of the following:

a) $f(x) = \dfrac{\sin x}{5x^2}$

b) $y = x\cos x^2$

c) $g(x) = \dfrac{\sqrt{x^2-5}}{e^x}$

Derivatives and Tangent Lines

Now that you have had a lot of practice finding derivatives, you can use what you learned to answer some questions about tangent lines to curves.

Example

Determine the slope of the tangent at $x = 3$ for the function $f(x) = \dfrac{4x^3}{\sin x}$.

Note: x is measured in radians.

Solution

First, find the derivative of $f(x)$ by rewriting $f(x)$ as $f(x) = 4x^3 \cdot (\sin x)^{-1}$.

Then, by the product rule, you have $f'(x) = (4x^3)'(\sin x)^{-1} + (4x^3)((\sin x)^{-1})'$.

You need to find the derivative of $4x^3$ and the derivative of $(\sin x)^{-1}$. The former is $12x^2$ by the power rule. The latter is $(-1)(\sin x)^{-2} \cdot (\sin x)'$ by the chain rule, which becomes $(-1)(\sin x)^{-2} \cdot \cos x$, since you know the derivative of sine is cosine. Putting all this together, you have $f'(x) = (12x^2)(\sin x)^{-1} + (4x^3)(-1)(\sin x)^{-2} \cdot \cos x$.

Therefore, the slope of the tangent line to the curve when $x = 3$ is

$$f'(3) = 12(3)^2 \cdot \frac{1}{\sin 3} - 4(3)^3 \frac{\cos 3}{(\sin 3)^2}$$

$$= 108 \cdot \frac{1}{0.1411} - 108 \cdot \frac{-0.9900}{(0.1411)^2} = 6135.8$$

Example

Determine all points on the curve $y = x^3 - 9x^2 + 14x - 7$ where the slope of the tangent line is -10.

Solution

The derivative of $y = x^3 - 9x^2 + 14x - 7$ is $y' = 3x^2 - 18x + 14$. If you're looking for points where the slope of the tangent line is -10, you need to set the derivative to -10 and solve:

$3x^2 - 18x + 14 = -10$

$3x^2 - 18x + 24 = 0$

$3(x^2 - 6x + 8) = 0$

$3(x - 2)(x - 4) = 0$

You must have $x = 2$ or $x = 4$. You can check both of these by plugging these values into the formula for the derivative:
$y'(2) = 3(2)^2 - 18(2) + 14 = -10$ and
$y'(4) = 3(4)^2 - 18(4) + 14 = -10$

The question asked for the points, so plug $x = 2$ and $x = 4$ into the original formula for y in order to find the corresponding y-coordinates.

For $x = 2$, you have $y = (2)^3 - 9(2)^2 + 14(2) - 7 = -7$.

For $x = 4$, you have $y = (4)^3 - 9(4)^2 + 14(4) - 7 = -31$.

The two points are (2, –7) and (4, –31).

Support Question
(do not send in for evaluation)

8. Determine all points on the curve $y = x^3 - 4x^2 + 5x - 6$ where the slope of the tangent line is 1.

Conclusion

In this lesson, you looked at the derivative of the cosine and sine functions and learned how to calculate the derivative of functions involving those trigonometric functions. In Lesson 8, you will look at the relationship between the derivative of a function and the graph of the function itself. You will also look at some real-world situations that you can solve using derivatives.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(14 marks)

16. Determine the slope of the tangent at $x = 0$ for the function $f(x) = \dfrac{\cos x}{1 - x}$. **(4 marks)**

17. Determine all points on the curve $y = x^2 + 4x - 3$ where the slope of the tangent line is -3. **(3 marks)**

18. Find the derivative of each of the following:

    a) $f(x) = \sqrt{x}(2x + 3)^2$ **(2 marks)**

    b) $g(x) = \dfrac{\sin x}{x^3 - 2x}$ **(2 marks)**

    c) $h(x) = 3e^{\sin(x+2)}$ **(3 marks)**

Now go on to Lesson 8. Do not submit your coursework to ILC until you have completed Unit 2 (Lessons 6 to 10).