MCV4U-A

Unit 2

Exploring Derivatives

December 6, 2010

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Introduction

The derivative can be used to help you understand a number of problems in real life that cannot be modelled by polynomial functions. For example, population growth is modelled by using exponential functions.

In this unit, you will explore the derivative of periodical functions, the logarithmic function, and the exponential function. You will also explore the connection between the derivative of a function and the behaviour of the function itself.

Overall Expectations

After completing this unit, you will be able to

- graph the derivatives of sinusoidal and exponential functions
- sketch the curve of a function using the numeric, graphical, and algebraic representations of a function and its derivatives
- sketch a reasonable graph for a polynomial function by identifying symmetries, finding maximum or minimum points and intervals of increase or decrease, and finding inflection points and intervals of concavity

MCV4U-A

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Exploring Exponential Functions and Their Derivatives

Introduction

In this lesson, you will explore the derivatives of exponential functions algebraically, and also learn the simple formula that allows you to find the derivative of any exponential function.

Estimated Hours for Completing This Lesson	
Exploring Exponential Functions	1.5
The Natural Exponential Function	1
The Natural Logarithm Function	1
Exponential Functions and the Chain Rule	0.5
Key Questions	1

What You Will Learn

After completing this lesson, you will be able to

- understand that exponential functions have a special property in that the derivative is a constant multiple of the original function
- explore the relationship between the logarithmic function and the exponential function
- calculate the derivative of any exponential function

Exploring Exponential Functions

An exponential function is any function of the form $f(x) = a^x$, where a is a constant. It requires a to be greater than 0, and typically forbids $a = 1$, since 1^x is an uninteresting function (its value is always 1). In this lesson, you'll focus on exponential functions where the base a is greater than 1. Such an exponential function is an increasing function and illustrates what's known as "exponential growth."

Here are some examples of the exponential functions you will be interested in:

- $f(x) = 1.1^x$
- $g(x) = 2^x$
- $h(x) = 2.5^x$
- $k(x) = 3^x$

Each of these is an increasing function, and their graphs have similar overall shapes but appear to increase at different rates.

Open "Lesson 6 Activity 1" on your course page at *ilc.org*. You will find a graph of the exponential function $y = m^x$, where the current value of m is set at 1.90.

By dragging the slider you can change the value of m, and watch the graph of $y = m^x$ change accordingly. Change the value of m to 4. Notice that a larger value of m results in a steeper graph.

By selecting and moving point A, you can observe the coordinates of different points on the graph of the exponential function. There is also a tangent line to the graph at point A, which moves when A moves. The slope of the tangent line is the same as the derivative of $y = m^x$ at point A.

Spend some time moving point A around for several different values of m. For any value of m greater than 1, you'll notice that the graph is increasing. As point A moves to the right, its y-coordinate increases. You'll also notice that the steepness of the graph increases as point A moves to the right.

Say $f(x)$ is an exponential function of the form $f(x) = m^x$ with $m = 1.90$. You know that when x gets bigger, the value of $f(x)$ gets bigger. You also know that when x gets bigger, the derivative $f'(x)$ also gets bigger. What is the relationship between $f'(x)$ and $f(x)$? You can illustrate the answer by creating a table that lists various values of $f(x)$ and $f'(x)$. Recall that $f'(x)$ is just the slope of the tangent to the curve at x. Move point A around and verify for yourself that you see values similar to those in the following table. (It might not be easy to get two or three decimal places of accuracy when you move the point around by hand, but just verify that these numbers are approximately correct.)

Point (*x*, *y*)	**Value of *x***	**Value of *f*(*x*) = 1.9^x**	**Value of *f*′(*x*)**
(–2, 0.277)	–2	0.277	0.178
(–1, 0.526)	–1	0.526	0.338
(0, 1.000)	0	1.000	0.642
(1, 1.900)	1	1.900	1.220
(2, 3.610)	2	3.610	2.317

At first glance, there might appear to be no obvious relationship between $f'(x)$ and $f(x)$, other than the vague observation that $f'(x)$ increases as $f(x)$ increases. It turns out that for exponential functions, however, there is a very simple relationship between $f'(x)$ and $f(x)$. Add another column to the table and find $\frac{f'(x)}{f(x)}$.

Point (*x*, *y*)	**Value of *x***	**Value of *f*(*x*) = 1.9^x**	**Value of *f*′(*x*)**	$\frac{f'(x)}{f(x)}$
(–2, 0.277)	–2	0.277	0.178	$\frac{0.178}{0.277} = 0.642$
(–1, 0.526)	–1	0.526	0.338	$\frac{0.338}{0.526} = 0.642$
(0, 1.000)	0	1.000	0.642	0.642
(1, 1.900)	1	1.900	1.220	0.642
(2, 3.610)	2	3.610	2.317	0.642

You can see that each number in the last column is 0.642. (The numbers you get might vary a bit.)

For each of the points in this table, you have $f'(x) = 0.642f(x)$.

Although it was not proven here algebraically, it is a general rule that if $f(x)$ is the exponential function $f(x) = 1.90^x$, then $f'(x) = 0.642f(x)$ for all real numbers x.

There's nothing special about using 1.90 as the base. A similar relationship exists for any exponential function, and you just have to replace 0.642 with a different constant. (What's not obvious is the relationship between the numbers 1.90 and 0.642. You'll see that a bit later in the lesson.) In general, you have the following rule.

> **Fact:** If $f(x) = a^x$ is any exponential function, where a is a constant greater than 1, you have $f'(x) = kf(x)$ for all real numbers x, where k is a constant.
>
> In other words, the derivative of an exponential function is proportional to the original function.

Exponential functions are the only functions for which the derivative is proportional to the original function. Exponential functions are often useful for modelling population growth, particularly for a population that has no shortage of resources, no natural predators, and no environmental catastrophes. Under these conditions, the rate of growth of the population tends to be proportional to the current population.

It is appropriate at this point to say a little bit about how you might try to find the derivative of an exponential function algebraically, using the definition of the derivative.

If $f(x) = a^x$, then using the first principle definition of the derivative:

$$f'(x) = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0}\frac{a^x(a^h - 1)}{h}$$

The common factor of a^x appearing in the numerator does not depend on h, so you can factor it out in front:

$$f'(x) = a^x \lim_{h\to 0}\frac{a^h - 1}{h}$$

Since a^x is just the original function $f(x)$, you have shown algebraically that $f'(x)$ is $f(x)$ times something. At this stage, you don't know any algebraic techniques for evaluating $\lim_{h\to 0}\frac{a^h-1}{h}$. Nevertheless, for any particular value of a, you can estimate this limit using numerical experimentation. For example, if a is 1.9 (so $f(x) = 1.9^x$ is the same exponential function you studied earlier in this lesson), then you can estimate $\lim_{h\to 0}\frac{a^h-1}{h}$ as follows:

$$\frac{1.9^{0.01}-1}{0.01} = 0.64392$$

$$\frac{1.9^{0.001}-1}{0.001} = 0.64206$$

$$\frac{1.9^{0.0001}-1}{0.0001} = 0.64187$$

This suggests that the true value of $\lim_{h\to 0}\frac{a^h-1}{h}$ is quite close to 0.642 if a is 1.9. This is consistent with the earlier stated fact that $f'(x) = 0.642\,f(x)$ when $f(x)$ is the exponential function $f(x) = 1.9^x$.

Support Question

(do not send in for evaluation)

1. Use "Lesson 6 Activity 1" to graph the following function by changing the value of m, complete the following tables, and then estimate the value of k in the statement $f'(x) = kf(x)$ for these exponential functions:

a) $f(x) = 1.7^x$

Point (x, y)	**Value of x**	**Value of $f(x) = 1.7^x$**	**Value of $f'(x)$**	$k = \frac{f'(x)}{f(x)}$
(0, 1)	0			
	1			
	2			
	3			
	4			
	5			

b) $g(x) = 2.3^x$

Point (x, y)	Value of x	Value of $g(x) = 2.3^x$	Value of $g'(x)$	$k = \frac{g'(x)}{g(x)}$
(0, 1)	0	1		
	1			
	2			
	3			
	4			

There are Suggested Answers to Support Questions at the end of this unit.

The Natural Exponential Function

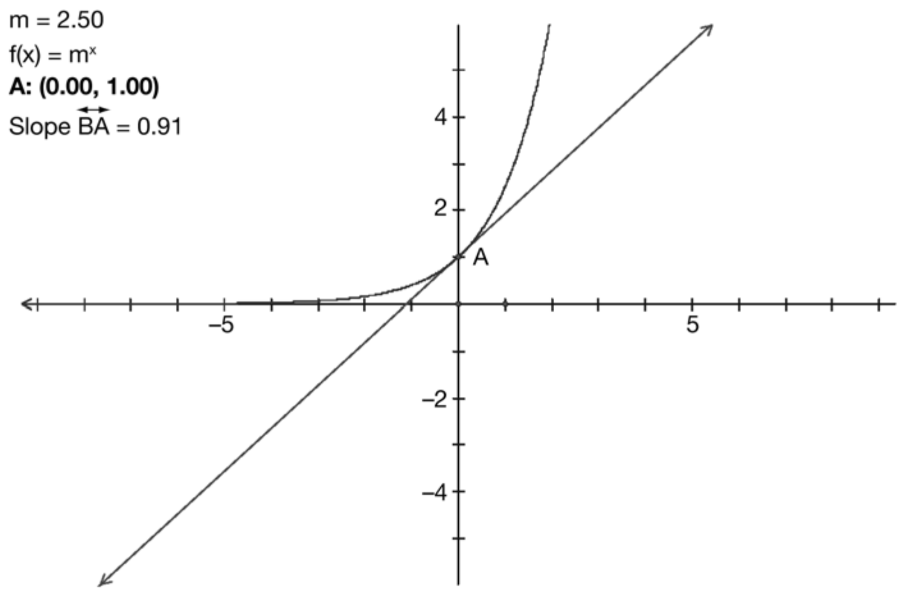
If $f(x) = a^x$ is any exponential function, where a is a real number greater than 1, then the graph of $f(x)$ goes through the point (0, 1), since $a^0 = 1$. You also know that $f'(x) = kf(x)$ for some constant k. Different values of a result in different values of k. For instance, you saw that when $a = 1.9$, you have $k = 0.642$.

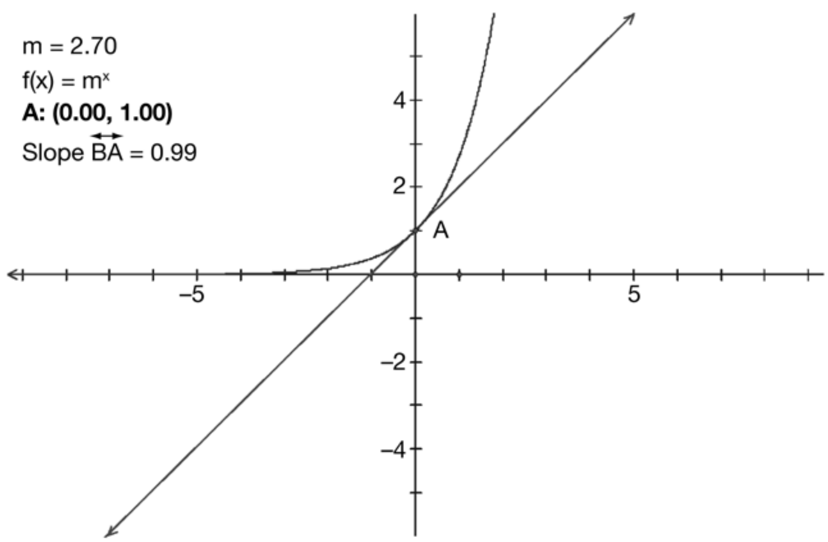
Since the relationship $f'(x) = kf(x)$ holds true for all values of x, then in particular it holds for $x = 0$:

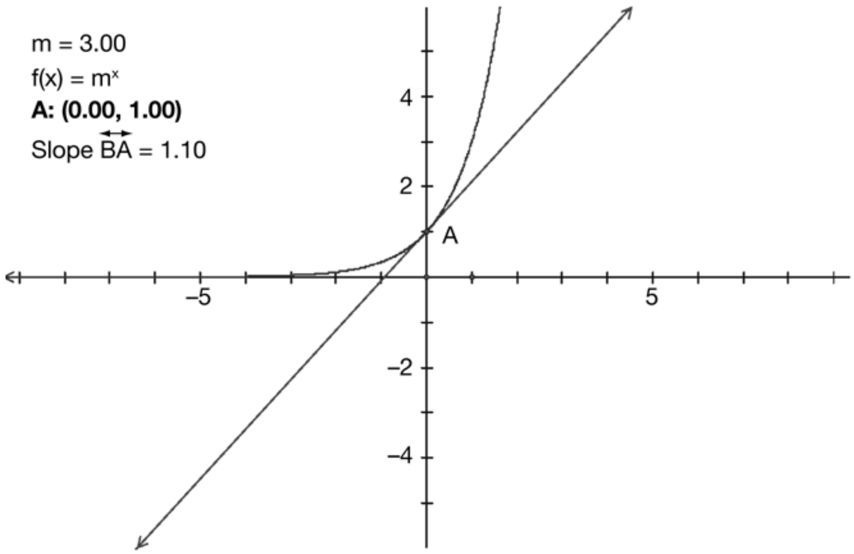
$f'(0) = kf(0)$, or more simply, $f'(0) = k \cdot 1 = k$.

That is, the constant k is equal to the slope of the exponential function at the point whose x-coordinate is 0 (the point (0, 1)).

When you experimented with different bases for the exponential function in “Lesson 6 Activity 1,” you found that increasing the base tends to result in a steeper graph. When the base is 1.9, the slope at the point (0, 1) is 0.642. If you try the bases 2.5, 2.7, and 3.0, you get pictures like the following graphs:


m = 2.50
f(x) = m^x
A: (0.00, 1.00)
Slope BA = 0.91
4
2
A
–5
5
–2
–4


m = 2.70
f(x) = m^x
A: (0.00, 1.00)
Slope BA = 0.99
4
2
A
–5
5
–2
–4



These graphs further illustrate that when you gradually increase the base, you also gradually increase the slope at the point (0, 1). You may also notice that something special happens if you choose the base to be slightly greater than 2.7: the slope at (0, 1) is 1.

Remember that the slope of the exponential function at the point (0, 1) is equal to the constant k in the relationship $f'(x) = kf(x)$. This means that the special base that results in a slope of 1 at (0, 1) also results in the special property that the function is equal to its own derivative.

The number that achieves this when you use it as the base is called e. A more accurate approximation of it is $e \approx 2.71828$. The number e is called Euler's constant, for the eighteenth-century Swiss mathematician Leonhard Euler. Most hand calculators have a button for the special exponential function e^x, which is sometimes called the natural exponential function.

To summarize:

The natural exponential function is the exponential function $f(x) = e^x$, whose base is the special number e, known as Euler's constant, which has an approximate value of 2.71828.

The natural exponential function has the special property that it is equal to its own derivative:

$f'(x) = f(x)$ or equivalently $f'(x) = e^x$.

In other words, every point on the graph of $y = e^x$ has the property that the value of the y-coordinate is the same as the slope of the tangent at that point.

The Natural Logarithm Function

For the natural exponential function $f(x) = e^x$, you have a very simple formula for the derivative. This allows you to answer questions easily about the rate of change of this function or the slope of the tangent line at a particular point.

What if you are asked similar questions about an exponential function $f(x) = a^x$ with a different base? You know the derivative of $f(x)$ has the form $f'(x) = ka^x$, but how do you find the value of k? It turns out that there is a simple relationship between k and a. In order to state it, you need to be familiar with the concept of the natural logarithm that you studied in a previous course (most likely MHF4U).

You should already be familiar with base-10 logarithms. In general, x is the base-10 logarithm of y, which is abbreviated as $x = \log y$, if you have $10^x = y$. This can be illustrated with some examples:

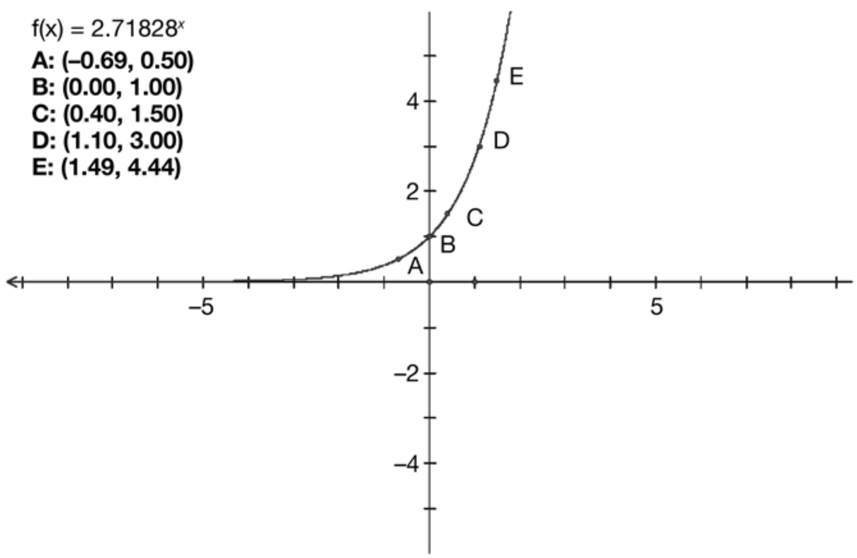
$$\log 100 = 2, \text{ since } 10^2 = 100$$

$$\log 1000 = 3, \text{ since } 10^3 = 1000$$

$$\log 0.001 = -3, \text{ since } 10^{-3} = 0.001$$

You can also define logarithms using Euler's constant $e \approx 2.71828$ as a base. Every positive number y can be written as e^x for some x, and that x is called the natural logarithm of y. The statement "x is the natural logarithm of y" is abbreviated $x = \ln y$, where "ln" stands for "logarithm" and "natural." This equation is often pronounced "x equals lawn y." Most hand calculators have a natural logarithm button labelled "ln."

You can illustrate natural logarithms a little better with the help of the graph of the function $y = e^x$. This is the graph of the exponential function whose base is the number $e \approx 2.71828$. Notice that the graph is an increasing curve where, for every possible positive y, there is an x such that $y = e^x$. For example, points on the graph are labelled where $y = 0.5$, $y = 1$, $y = 1.5$, $y = 3$, and $y = 4.44$.



Since $(-0.69, 0.5)$ is a point on the graph, you have $e^{-0.69} = 0.5$ and $\ln 0.5 = -0.69$.

Since $(0, 1)$ is a point on the graph, you have $e^0 = 1$ and $\ln 1 = 0$.

Since $(0.4, 1.5)$ is a point on the graph, you have $e^{0.4} = 1.5$ and $\ln 1.5 = 0.4$.

Since $(1.1, 3)$ is a point on the graph, you have $e^{1.1} = 3$ and $\ln 3 = 1.1$.

Since (1.49, 4.44) is a point on the graph, you have $e^{1.49} = 4.44$ and $\ln 4.44 = 1.49$.

You can use your calculator to find ln 0.5, ln 4.44, and so on, to verify that you get results close to those listed. (Due to rounding, your results might not be exactly identical.)

Much more could be said about natural logarithms. For now, being familiar with the concept of the natural logarithm allows you to state the rule for finding the derivative of an exponential function with any base. It also means it's possible to provide an answer to the earlier question: "What is the relationship between 0.642 and 1.9?"

Here's the answer:

Fact: If $f(x) = a^x$ is any exponential function, where a is a positive constant, then the derivative of $f(x)$ is given by the formula $f'(x) = a^x \ln a$.

For example, since $\ln 1.9 \approx 0.642$, the derivative of $f(x) = 1.9^x$ is $f'(x) = 1.9^x \ln 1.9 \approx 0.642 \cdot 1.9^x$.

As another example, the derivative of $g(x) = 5^x$ is $g'(x) = 5^x \ln 5 \approx 1.609 \cdot 5^x$.

Earlier in this lesson, it was stated that any exponential function $f(x)$ obeys the rule that $f'(x) = kf(x)$ for some constant k. Now you know the rule for finding the value of k: $k = \ln a$ where a is the base of the exponential function.

At this stage, you don't have the techniques to prove this rule algebraically, but you can verify that the rule is consistent with experience by experimenting with "Lesson 6 Activity 1." For instance, if you change the base of the exponential function to 5, the slope at the point (0, 1) becomes approximately 1.609, suggesting that 1.609 is indeed the appropriate value of k for the exponential function $f(x) = 5^x$.

Example

Determine the slope of the tangent to the curve $f(x) = 5^x$ at the point (1, 5).

Solution

$f'(x) = a^x \ln a = 5^x \ln(5)$

At $x = 1$, $f'(1) = 5 \ln(5)$

The slope of the tangent is $5 \ln(5) \approx 8.04$

Support Question
(do not send in for evaluation)

2. Determine the instantaneous rate of change of the function $f(x) = 3^x$ at the point (2, 9).

Exponential Functions and the Chain Rule

Suppose you want to find the derivative of a function involving a polynomial expression in the exponent. For example, $f(x) = 3^{x^2-5x}$. To do this, you would use the chain rule applied to exponential functions.

Chain rule for exponential functions:

The derivative of the function $f(x) = a^{g(x)}$, where a is a positive constant, is given by $f'(x) = a^{g(x)} \ln a\, g'(x)$.

Examples

Find the derivative of each of the following:

a) $f(x) = 3^{x^2-5x}$

b) $g(x) = x^2e^{4x^3}$

c) $y = \dfrac{2x}{e^{x^2}}$

Solutions

a) Use the chain rule, $f'(x) = a^{g(x)} \ln a g'(x)$

$f'(x) = 3^{x^2-5x} \cdot \ln 3 \cdot (2x-5)$

b) Use the product rule:

$g'(x) = (x^2)'e^{4x^3} + x^2(e^{4x^3})'$

$= 2x \cdot e^{4x^3} + x^2 \cdot e^{4x^3} \cdot 12x^2$

$= 2xe^{4x^3} + 12x^4e^{4x^3}$

c) Rewrite y as $y = 2xe^{-x^2}$, then use the product rule:

$y' = (2x)'e^{-x^2} + 2x(e^{-x^2})'$

$= 2e^{-x^2} + 2x(-x^2)'e^{-x^2}$

$= 2e^{-x^2} + (2x)(-2x)e^{-x^2}$

$= 2e^{-x^2} - 4x^2e^{-x^2}$

Support Questions
(do not send in for evaluation)

3. Find the derivative of each of the following:

   a) $y = e^{3x} + 2$

   b) $y = 3x^2 - e^{x^2}$

   c) $f(x) = 2^x - x^2$

   d) $g(x) = 6^{x^2 - 3x + 4}$

4. Find the equation of the tangent to the function $f(x) = 3^{x^2+1} - x^2$ at $(1, 8)$.

Conclusion

In this lesson, you learned about exponential functions and their derivatives. Exponential functions are the only functions with the property that their derivative is proportional to the function itself. In future lessons, you will use what you learned in this lesson to solve problems arising from real-world situations.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(10 marks)

13. If $f(x) = e^{3x^2+x}$, find $f'(2)$. **(2 marks)**

14. Find the slope of the tangent to the function $f(x) = 2^{x^2+3x}$ when $x = 3$. **(2 marks)**

15. Differentiate the following:

    a) $y = xe^{2x}$ **(1 mark)**

    b) $f(x) = 3^x + x^3$ **(1 mark)**

    c) $y = e^{-(2x+5)}$ **(1 mark)**

    d) $y = e^{3x^2-5x+7}$ **(1 mark)**

    e) Explain in your own words why the derivative of $f(x) = e^x$ is $f'(x) = e^x$. **(2 marks)**

Now go on to Lesson 7. Do not submit your coursework to ILC until you have completed Unit 2 (Lessons 6 to 10).