MCV4U-A

5

Properties of the Derivative

Introduction

At this point, you may be thinking that finding the derivative of a polynomial function is a long and complicated process. Haven't mathematicians discovered a simpler way to do this?

In this lesson, you will continue to explore, algebraically, the derivatives of polynomial functions. In addition, you will learn, verify, and apply various rules to calculate the derivative of polynomial functions.

Estimated Hours for Completing This Lesson	
Properties of Derivatives	2.5
Equation of a Tangent to a Curve	1
Key Questions	1.5

What You Will Learn

After completing this lesson, you will be able to

- apply the following rules to find the derivative: the power rule, the constant rule, the constant multiple rule, the sum and difference rule, the product rule, the chain rule
- find the equation of the tangent line to a curve

Properties of Derivatives

In Lesson 4, you used $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ to calculate the derivative of a function. This method is time-consuming. In this lesson, you will discover rules to help you find the derivative without calculating the limit.

The Power Rule

Recall from Lesson 4 that you used $\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ to calculate the derivative of x^2 and x^3. In this lesson, you will discover a rule that you can apply to find the derivative of $f(x) = x^n$. The following table summarizes some derivatives of $f(x) = x^n$:

Function	Derivative
$f(x) = x^2$	$f'(x) = 2x$
$f(x) = x^3$	$f'(x) = 3x^2$
$f(x) = x^4$	$f'(x) = 4x^3$

Do you see a pattern? The degree of the derivative is one less than the original function. What else do you notice? The coefficient of the derivative is the power of the original function.

Therefore, the power rule states that for $f(x) = x^n$, when n is an integer number, then $f'(x) = nx^{n-1}$.

Does the power rule work if n is not an integer? In Lesson 4, you saw that the derivative of $\sqrt{x}$ is $\dfrac{1}{2\sqrt{x}}$. Another way to write $\sqrt{x}$ is $x^{\frac{1}{2}}$.

Use the power rule formula for $n = \frac{1}{2}$ and see if it works:

$f'(x) = nx^{n-1}$

$$\begin{aligned} f'(x) &= \frac{1}{2}x^{\frac{1}{2}-1} \\ &= \frac{1}{2}x^{\frac{-1}{2}} \\ &= \frac{1}{2x^{\frac{1}{2}}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

The power rule works for this example where n is not an integer. While this is not a proof, it is known that the power rule works for all real numbers.

The Power Rule

$f(x) = x^n$, n is any real number, then $f'(x) = nx^{n-1}$

Examples

Find the derivative of each of the following functions using the power rule:

a) $f(x) = x^6$

b) $f(x) = x^{12}$

c) $f(x) = \frac{1}{x^4}$

d) $f(x) = \sqrt[\frac{5}{2}]{x^3}$

Solutions

a) Use the power rule where $n = 6$:

$f'(x) = nx^{n-1}$

$f'(x) = 6x^{6-1}$

$f'(x) = 6x^5$

b) $f'(x) = 12x^{12-1}$

$f'(x) = 12x^{11}$

c) Recall that $\dfrac{1}{x^m}$ can be written as x^{-m}. Rewrite $f(x)$:

$$f(x) = \frac{1}{x^4} = x^{-4}$$

$$\begin{aligned} f'(x) &= -4x^{-4-1} \\ &= -4x^{-5} \\ &= \frac{-4}{x^5} \end{aligned}$$

d) Rewrite $f(x)$:

$$\begin{aligned} f(x) &= \sqrt[\frac{5}{2}]{x^3} \\ &= (x^3)^{\frac{2}{5}} \\ &= x^{\frac{6}{5}} \end{aligned}$$

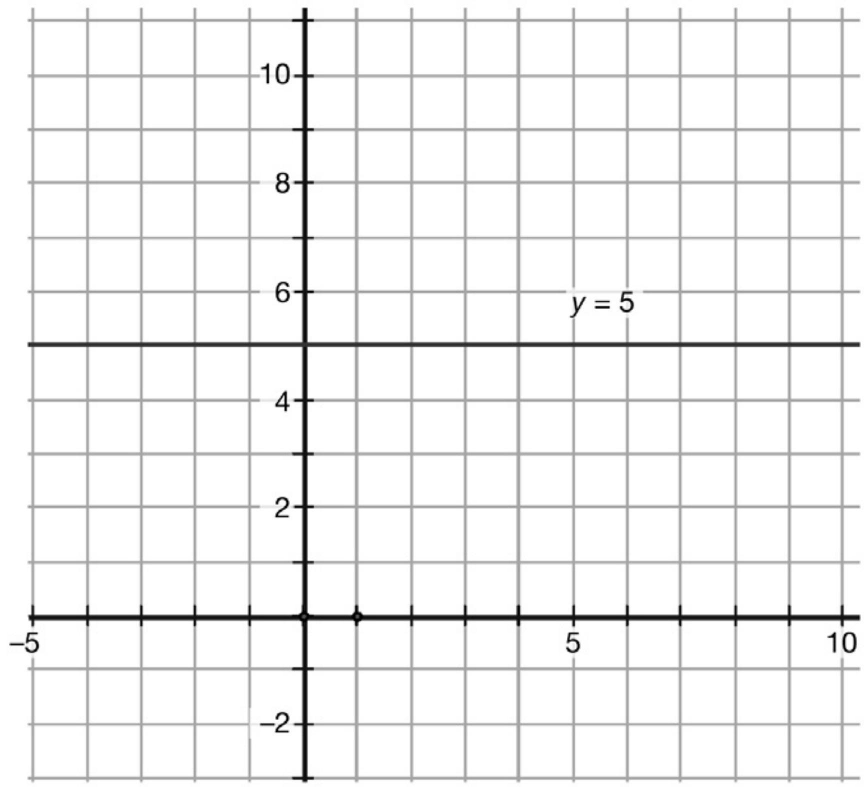
Apply the power rule to $f(x) = x^{\frac{6}{5}}$:

$$\begin{aligned} f'(x) &= \frac{6}{5}x^{\frac{6}{5}-1} \\ &= \frac{6}{5}x^{\frac{1}{5}} \end{aligned}$$

The Constant Rule

The constant rule gives the derivative of a constant function. A constant function is a function with a constant value for all x, for example, $f(x) = 5$.

Recall that the derivative function gives you the slope of the tangent at a point that is the same as the instantaneous rate of change of the function at that given point.



The rate of change of a constant function is 0, since the function itself does not change as x varies. You can conclude that the derivative of a constant function is 0.

The Constant Rule

$f(x) = c$, c is any real number, then $f'(x) = 0$

The Constant Multiple Rule

Suppose you know the derivative of $f(x)$ and you wish to calculate the derivative of $g(x) = cf(x)$. Use the first principle to show that $g'(x) = cf'(x)$:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}\\
&= \lim_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right]\\
&= c\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]
\end{aligned}$$

What do you notice? $\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$ is the derivative of $f(x)$.

You can conclude that $g'(x) = cf'(x)$.

The Constant Multiple Rule

$f(x) = cg(x)$, c is any real number, then $f'(x) = cg'(x)$

Examples

Find the derivative of each of the following functions:

a) $f(x) = 3x^2$

b) $y = -4x^2$

c) $f(x) = 2\sqrt{x}$

Solutions

a)
$$\begin{aligned} f'(x) &= (3x^2)' \\ &= 3(x^2)' \\ &= 3(2x) \\ &= 6x \end{aligned}$$

b)
$$\begin{aligned} y' &= (-4x^2)' \\ &= -4(x^2)' \\ &= -4(2x) \\ &= -8x \end{aligned}$$

c)
$$\begin{aligned} f'(x) &= (2\sqrt{x})' \\ &= 2(\sqrt{x})' \\ &= 2(x^{\frac{1}{2}})' \\ &= 2 \times \frac{1}{2} x^{\frac{1}{2}-1} \\ &= x^{-\frac{1}{2}} \end{aligned}$$

Support Question
(do not send in for evaluation)

17. Find the derivative of each function:

    a) $y = x^{456}$

    b) $y = -3.12$

    c) $f(x) = -3x^2$

    d) $y = \sqrt[3]{x}$

    e) $y = \sqrt{4x^3}$

There are Suggested Answers to Support Questions at the end of this unit.

The Sum and Difference Rules

The sum rule gives you a way to calculate the derivative of a sum of two functions. The rule states that the derivative of a sum is the sum of derivatives. This is proven by using the first principle:

$F(x) = f(x) + g(x)$

$$\begin{aligned}
\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right]\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x)
\end{aligned}$$

Similarly, the difference rule states that if $F(x) = f(x) - g(x)$ then $F'(x) = f'(x) - g'(x)$.

The Sum Rule

$F(x) = f(x) + g(x)$ then $F'(x) = f'(x) + g'(x)$

The Difference Rule

$F(x) = f(x) - g(x)$ then $F'(x) = f'(x) - g'(x)$

In addition, you can extend the sum rule to find the derivative of the sum of any number of functions. For example,
$(f(x) + g(x) + h(x))' = f'(x) + g'(x) + h'(x)$

Examples

Differentiate the following functions:

a) $f(x) = 3x^2 + 4x$

b) $f(x) = 2x^4 + 3x^2 - 3$

c) $y = -x^6 + 3x^4 - 5x$

Solutions

Use the sum or difference rule (or both) of the derivative:

a) $$\begin{aligned} f'(x) &= (3x^2)' + (4x)' \\ &= 3(x^2)' + 4(x)' \\ &= 3 \times 2x + 4 \\ &= 6x + 4 \end{aligned}$$

b) $$\begin{aligned} f'(x) &= (2x^4)' + (3x^2)' - (3)' \\ &= 2 \times 4x^3 + 3 \times 2x - 0 \\ &= 8x^3 + 6x \end{aligned}$$

c) $$\begin{aligned} f'(x) &= (-x^6)' + (3x^4)' - (5x)' \\ &= -6x^5 + 12x^3 - 5 \end{aligned}$$

Examples

Find the value of each derivative at the given value:

a) $f(x) = x^3 - 5x + 1$; $x = -1$

b) $f(x) = 3x^4 - 3x + 1$; $x = 3$

Solutions

a) First find the derivative:

$$\begin{aligned} f'(x) &= (x^3)' - (5x)' + (1)' \\ &= 3x^2 - 5 \end{aligned}$$

Next, substitute the given value of x:

$$\begin{aligned} f'(-1) &= 3(-1)^2 - 5 \\ &= 3 - 5 \\ &= -2 \end{aligned}$$

b) $$\begin{aligned} f'(x) &= 12x^3 - 3 \\ f'(3) &= 12(3)^3 - 3 \\ &= 324 - 3 \\ &= 321 \end{aligned}$$

Support Questions
(do not send in for evaluation)

18. Find the derivative of each function:

a) $h(x) = 3x^5 - 4x^3 + 3$

b) $y = 3x^{1.5} - 3x$

19. Find the slope of the tangent to the curve at x for each of the following functions:

a) $y = x^3 - 2x^2$ at $x = 2$

b) $f(x) = (3x)^2 - 5x$ at $x = 0$

The Product Rule

The product rule helps determine the derivative of the product of two functions. You might think that the derivative of the product is the product of the derivatives. This is not correct! Here is an explanation:

$F(x) = f(x)g(x)$ where $f(x)) = x^2$ and $g(x) = x^3$

$F(x) = (x^2)(x^3) = x^5$

Using the power rule, $F'(x) = 5x^4$

But $f'(x)g'(x) = (2x)(3x^2) = 6x^3$

Therefore, you can conclude that in general $(f(x)g(x))' \neq f'(x)g'(x)$.

The Product Rule
$F(x) = f(x)g(x)$ then $F'(x) = f(x)g'(x) + f'(x)g(x)$

It may help if you use words to describe the equation:

The derivative of a product of functions is the first function times the derivative of the second function plus the derivative of the first function times the second function.

Example

a) Find the derivative of $f(x) = (2x^2 + 2x)(3x^4 + 2)$ using the product rule.

b) Verify your answer to (a) by expanding the original question and finding the derivative of the polynomial.

Solution

a) $f(x)$ is the product of two functions, so use the product rule:

$$\begin{aligned} f'(x) &= (2x^2 + 2x)(3x^4 + 2)' + (2x^2 + 2x)'(3x^4 + 2) \\ &= (2x^2 + 2x)(12x^3) + (4x + 2)(3x^4 + 2) \\ &= 24x^5 + 24x^4 + 12x^5 + 8x + 6x^4 + 4 \\ &= 36x^5 + 30x^4 + 8x + 4 \end{aligned}$$

b) Expand the function $f(x) = (2x^2 + 2x)(3x^4 + 2)$:

$$\begin{aligned} f(x) &= (2x^2 + 2x)(3x^4 + 2) \\ &= 6x^6 + 4x^2 + 6x^5 + 4x \\ &= 6x^6 + 6x^5 + 4x^2 + 4x \end{aligned}$$

Find the derivative using the sum rule:

$$f'(x) = 36x^5 + 30x^4 + 8x + 4$$

Examples

Find the derivative of the following functions:

a) $h(x) = \sqrt{x}(x^2 + 3)$

b) $y = \dfrac{2x - 1}{x^2}$

Solutions

a) Recall that $\sqrt{x} = x^{\frac{1}{2}}$, so you can write $h(x) = x^{\frac{1}{2}}(x^2 + 3)$:

$$\begin{aligned} h'(x) &= x^{\frac{1}{2}}(x^2 + 3)' + (x^{\frac{1}{2}})'(x^2 + 3) \\ &= x^{\frac{1}{2}}(2x) + \frac{1}{2}x^{\frac{1}{2}-1}(x^2 + 3) \\ &= 2x^{\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}(x^2 + 3) \\ &= 2x^{\frac{3}{2}} + \frac{x^2 + 3}{2x^{\frac{1}{2}}} \end{aligned}$$

b) $y = \dfrac{2x - 1}{x^2}$

You can write this as a product, since dividing by x^2 is the same as multiplying by $\dfrac{1}{x^2}$, but $\dfrac{1}{x^2} = x^{-2}$:

$$y = \frac{2x - 1}{x^2} = (2x - 1)x^{-2}$$

To find the derivative, use the product rule:

$$\begin{aligned} y' &= (2x-1)(x^{-2})' + (2x-1)'(x^{-2}) \\ &= (2x-1)(-2x^{-3}) + 2x^{-2} \\ &= -4x^{-2} + 2x^{-3} + 2x^{-2} \\ &= -2x^{-2} + 2x^{-3} \end{aligned}$$

Support Question
(do not send in for evaluation)

20. Differentiate each of the following using the product rule. You do not need to simplify your answer.

a) $y = (x^2 - 3x)(x^4 - 4x^3 - 5)$

b) $h(x) = (x^4 - 4x^3 + x)(2x^6 - 5x^4 + 8x^2 - x)$

c) $y = (\sqrt{x} - x^2 + x)(x^3 - 2x^2 + x)$

d) $h(t) = \dfrac{t^3 - t + 1}{t^2}$

The Chain Rule for Powers of Polynomials

The chain rule helps you find the derivative of a function of the form $F(x) = (f(x))^n$, where $f(x)$ is a polynomial function. $f(x) = (3x^2 - 4x)^3$ is an example of such a function.

To find the derivative of such a function, multiply n by the derivative of $f(x)$ and $f(x)$ to the power $n - 1$.

The Chain Rule for Powers of Polynomials

$F(x) = (f(x))^n$, then $F'(x) = nf'(x)f(x)^{n-1}$

Examples

Differentiate the following functions:

a) $y = (x^2)^5$

b) $F(x) = (3x^2 - 4x)^3$

c) $y = \dfrac{1}{\sqrt[4]{x^2 - 4}}$ (write answer in radical form)

d) $y = \sqrt{x - \sqrt{x + 1}}$

Solutions

a)
$$\begin{aligned} y' &= 5(x^2)'(x^2)^4 \\ &= (5 \times 2x)(x^2)^4 \\ &= (10x)(x^8) \\ &= 10x^9 \end{aligned}$$

Note: You can verify that this solution is correct by simplifying $y = (x^2)^5 = x^{10}$ and using the standard power rule to take the derivative, which is $10x^9$.

b)
$$\begin{aligned} F'(x) &= 3(3x^2 - 4x)'(3x^2 - 4x)^{3-1} \\ &= 3(6x - 4)(3x^2 - 4x)^2 \end{aligned}$$

c) Write y in exponent form, then apply the chain rule:

$$\begin{aligned} y &= \frac{1}{\sqrt[4]{x^2 - 4}} \\ &= (x^2 - 4)^{-\frac{1}{4}} \end{aligned}$$

$$\begin{aligned} y' &= -\frac{1}{4}(x^2 - 4)'(x^2 - 4)^{-\frac{1}{4}-1} \\ &= -\frac{1}{4}(2x)(x^2 - 4)^{-\frac{5}{4}} \\ &= \frac{-x}{2\sqrt[4]{(x^2 - 4)^5}} \end{aligned}$$

d) Remember that $\sqrt{x} = x^{\frac{1}{2}}$, so you can rewrite:

$$y = \sqrt{x - \sqrt{x+1}}$$
$$= (x - (x+1)^{\frac{1}{2}})^{\frac{1}{2}}$$

$$y' = \frac{1}{2}(x - (x+1)^{\frac{1}{2}})'(x - (x+1)^{\frac{1}{2}})^{\frac{1}{2}-1}$$

Note: $(x - (x+1)^{\frac{1}{2}})' = 1 - \frac{1}{2}(x+1)^{-\frac{1}{2}}$

$$y' = \frac{1}{2}(1 - \frac{1}{2}(x+1)^{-\frac{1}{2}})(x - (x+1)^{\frac{1}{2}})^{-\frac{1}{2}}$$

Support Question
(do not send in for evaluation)

21. Differentiate the following functions. You do not need to simplify the answers.

a) $y = (x^3 - 4x + 1)^4$

b) $f(x) = x\sqrt{x^3 + 2x}$ (write $f'(x)$ in radical form)

Equation of a Tangent to a Curve

The derivative of a function at a point A is the slope of the tangent to the curve at A.

Suppose you want to find the equation of the tangent to the function $f(x) = x^3 - 5x + 1$ at $x = -1$.

First you find the derivative, $f'(x) = 3x^2 - 5$, and then to find the slope of the tangent you calculate $f'(-1) = 3(-1)^2 - 5 = -2$.

To find the equation of a line, you usually need a point and the slope of the line. You have both in this example: although you have only the x-coordinate of the point, since the point is on the curve you can calculate its y-coordinate by substituting $x = -1$ into the function:

$$
\begin{aligned}
f(-1) &= (-1)^3 - 5(-1) + 1 \\
&= -1 + 5 + 1 \\
&= 5
\end{aligned}
$$

Now, you need to find the y-intercept:

$$
\begin{aligned}
y &= mx + b \\
5 &= (-2)(-1) + b \\
5 &= 2 + b \\
5 - 2 &= b \\
3 &= b
\end{aligned}
$$

The equation of the tangent to $f(x)$ at $x = -1$ is $y = -2x + 3$.

Examples

Find the equation of the tangent line to the curve of each function at the given x.

a) $y = 3x^3 - 2x + 3$ at $x = 1$

b) $y = 3x^2 - 6x$ at $x = 1$

Solutions

a) The y-coordinate of the point on the curve at $x = 1$ is

$$\begin{aligned} 3(1)^3 - 2(1) + 3 &= 3 - 2 + 3 \\ &= 4 \end{aligned}$$

The point on the curve is (1, 4).

To find the slope, calculate the derivative at $x = 1$:

$$\begin{aligned} y' &= (3x^3)' - (2x)' + (3)' \\ &= 9x^2 - 2 \end{aligned}$$

The slope of the tangent is the derivative at $x = 1$, $9(1)^2 - 2 = 7$.

Using the slope of the tangent, the equation of the line is $y = 7x + b$.

To find the y-intercept, substitute the coordinates of the point (1, 4) into $y = 7x + b$.

$4 = 7(1) + b$

$4 - 7 = b$

$b = -3$

The equation of the tangent is $y = 7x - 3$.

b) The y-coordinate of the point on the curve at $x = 1$ is $3(1)^2 - 6(1) = -3$

The point on the curve is (1, –3).

To find the slope, calculate the derivative at $x = 1$:
$y' = 6x - 6$

The slope of the tangent at $x = 1$ is $6(1) - 6 = 0$.

The equation of the tangent is $y = -3$.

What if you are asked to find the equation of the tangent that has a specific direction? Here is an example.

Example

Find the equation of the tangent to $y = 2x^2 + 4x - 1$ that is parallel to the line $y = 8x + 3$.

Solution

To find the equation of a line, you need a point and a slope. Two parallel lines have the same slope. The slope of $y = 8x + 3$ is 8, and hence the slope of the tangent is 8.

Now you need only find the point on the curve where the derivative is 8.

$y' = 4x + 4$

$8 = 4x + 4$

$8 - 4 = 4x$

$4 = 4x$

$x = 1$

To solve for y, substitute the value of x into the function:

$y = 2(1)^2 + 4(1) - 1$

$\quad = 5$

The point on the curve is (1, 5).

Now use the coordinates of (1, 5) to find the y-intercept.

$y = 8x + \mathrm{b}$

$5 = 8(1) + b$

$b = -3$

The equation of the tangent is $y = 8x - 3$

Support Questions

(do not send in for evaluation)

22. Find the equation of the tangent to the curve at the given x:

    a) $y = x^4 - 3x + 1$ at $x = 0$

    b) $y = 3x^4 - 3x^3 + 4$ at $x = 1$

23. Find the point at which the slope of the tangent is m and find the equation of the tangent. There may be more than one tangent.

    a) $y = x^3 - 2x^2 + 2x - 1$, $m = 1$

    b) $y = -2x^2 + 4x$, $m = 3$

Before you start the Key Questions, here's a review of the rules introduced in this lesson:

The Power Rule

$f(x) = x^n$, n is any real number, then $f'(x) = nx^{n-1}$

The Constant Rule

$f(x) = c$, c is any real number, then $f'(x) = 0$

The Constant Multiple Rule

$f(x) = cg(x)$, c is any real number, then $f'(x) = cg'(x)$

The Sum Rule

$F(x) = f(x) + g(x)$ then $F'(x) = f'(x) + g'(x)$

The Difference Rule

$F(x) = f(x) - g(x)$ then $F'(x) = f'(x) - g'(x)$

The Product Rule

$F(x) = f(x)g(x)$ then $F'(x) = f(x)g'(x) + f'(x)g(x)$

The Chain Rule for Polynomials

$F(x) = (f(x))^n$, then $F'(x) = nf'(x)f(x)^{n-1}$

Conclusion

In this lesson, you learned how to find the derivative of polynomial functions using a variety of tools. In subsequent lessons, you will learn to apply these rules to find the derivative of different kinds of functions, including trigonometric and exponential. You will also look at a number of applications of derivatives in real-world situations.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(26 marks)

10. Differentiate each of the following functions:

    a) $f(x) = x^4 - 2x^2 + x$ **(1 mark)**

    b) $f(x) = (x^2 - 3x)^2$ **(2 marks)**

    c) $f(x) = (x^2 + 2)(2x^3 - 5x^2 + 4x)$ **(3 marks)**

    d) $f(x) = \sqrt{x} - \sqrt[3]{x}$ **(4 marks)**

    e) $f(x) = \dfrac{3x}{x^2 + 4}$ **(4 marks)**

11. Find the points on the curve $y = \dfrac{2}{3x - 2}$ where the tangent is parallel to the line $y = -\dfrac{3}{2}x - 1$. **(5 marks)**

12. Find the equation of the tangent to $y = x^2 - 3x - 4$ that is parallel to the line $y = 7x + 3$. **(7 marks)**

This is the last lesson in Unit 1. When you are finished, do the Reflection for Unit 1. Follow any other instructions you have received from ILC about submitting your coursework, then send it to ILC. A teacher will mark your work and you will receive your results online.