MCV4U-A

4

Derivative of a Polynomial Function

Introduction

The derivative is one of the main foundations of calculus. In simplified terms, it can be thought of as how much a quantity is changing at some given point. For example, suppose you are blowing up a balloon and want to know how fast the volume is changing. The volume of the balloon depends on the radius. The derivative of the volume would tell you how fast the volume is changing as the radius changes.

In this lesson, you will explore the derivative of polynomial functions as the function that gives you the instantaneous rate of change at a given point. You will learn how to calculate the derivative of a given polynomial using the first principle definition of the derivative.

Estimated Hours for Completing This Lesson	
Sketching the Derivative	2.5
First Principle Definition of the Derivative	1.5
Key Questions	1

What You Will Learn

After completing this lesson, you will be able to

- sketch the graph of the derivative of a quadratic, cubic, and quartic function
- determine the derivative of a given polynomial function using the first principle definition

Sketching the Derivative

You will start by investigating the slope of the tangent to a polynomial function. Recall that a polynomial function is a function that has the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ where $n \geq 0$.

The degree of a polynomial function is the highest exponent of the variables. For example, $f(x) = 3x^4 - 3x + 1$ is a polynomial function of degree 4. On the other hand, $g(x) = \dfrac{3x - 5}{x^2}$ isn't a polynomial function since the function has a different form.

Instantaneous Rate of Change of a Quadratic Function

Open "Lesson 4 Activity 1" on your course page at *ilc.org*.

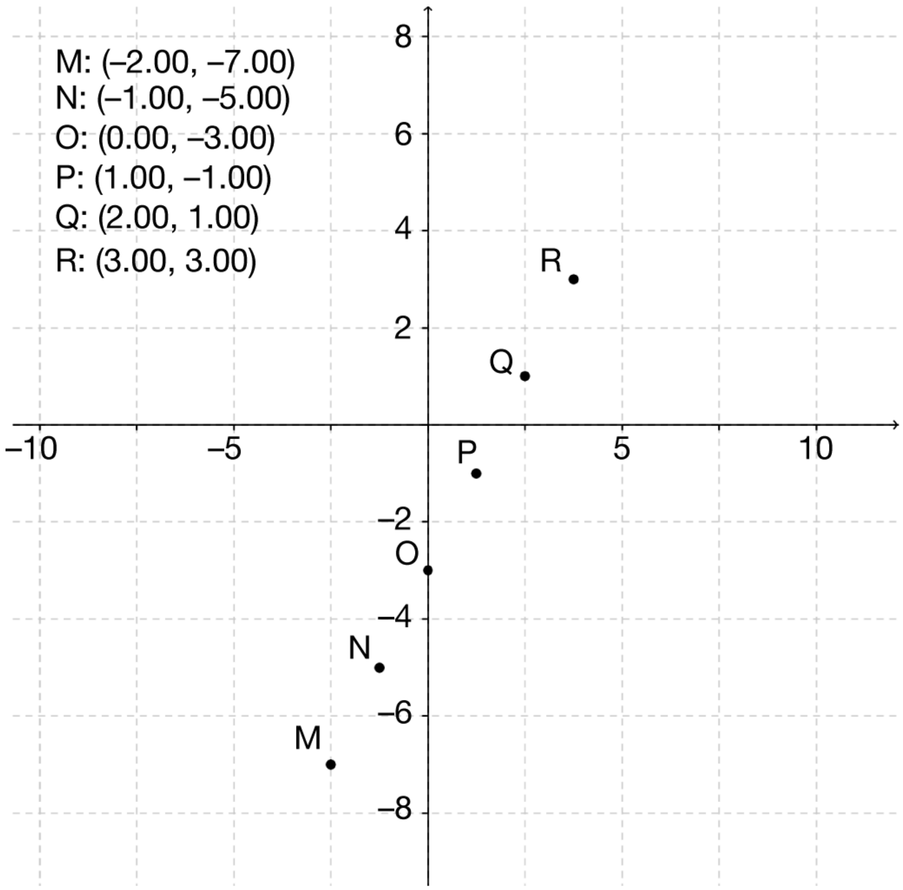
The applet includes a sketch of the function $f(x) = x^2 - 3x - 1$, a point on the curve labelled A, and a tangent to the curve at A.

As you move point A, check to see that you get the same values for the slope of the tangents that are listed in the following table. (Moving the point around by hand, it might not be easy to get two or three decimal places of accuracy, but you can verify that these numbers are approximately correct):

x	**Slope of the tangent**	**Ordered pair (*x*, slope at *x*)**
–2	–7	(–2, –7)
–1	–5	(–1, –5)
0	–3	(0, –3)
1	–1	(1, –1)
2	1	(2, 1)
3	3	(3, 3)

Plot the ordered pairs in the table on graphing paper to check if a pattern exists.

Your graph should look like this:



You can see on the graph that the relationship is linear. You can confirm this by calculating the first differences of the slopes:

x	Slope of the tangent	First difference
–2	–7	
–1	–5	$-5 - (-7) = -5 + 7 = 2$
0	–3	$(-3) - (-5) = 2$
1	–1	$(-1) - (-3) = 2$
2	1	$1 - (-1) = 2$
3	3	$3 - 1 = 2$

Since all the first differences are the same, the graph of the derivative is a line and hence its equation is also a line.

In general, the derivative of a function is the equation that gives the slope of tangents at any point on the curve of the function. A number of notations are used for the derivative of a function $f(x)$.

The most common are $f'(x), \dfrac{df}{dx}, y', \dfrac{dy}{dx}$.

Example

Open "Lesson 4 Activity 2" on your course page at *ilc.org*.

Use the graph $f(x) = x^3 - 3x + 1$ to complete the following table:

x	**Slope of the tangent**	**Ordered pair (x, slope at x)**
–2		
–1		
0		
1		
2		

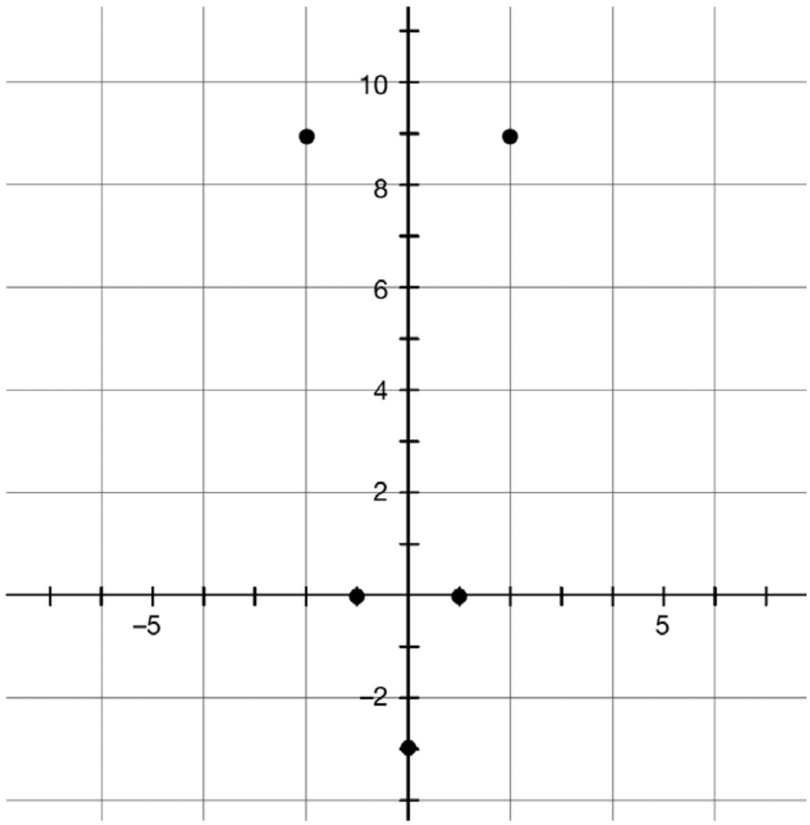
The applet includes a sketch of the function $f(x) = x^3 - 3x + 1$, a point, A, on the curve, and a tangent to the curve at A. Click and drag point A. You will notice that the slope of the tangent changes accordingly.

Solution

When you collect the values of the slope, you should get the following table:

x	**Slope of the tangent**	**Ordered pair (x, slope at x)**
–2	9	(–2, 9)
–1	0	(–1, 0)
0	–3	(0, –3)
1	0	(1, 0)
2	9	(2, 9)

When you plot these ordered pairs, you get the following graph:



In this example, the points seem to follow the shape of a parabola (recall that a parabola is the graph of a quadratic function). You can confirm this by calculating the second difference, which is the difference of the first differences:

x	Slope of the tangent	First difference	Second difference
–2	9		
–1	0	0 – 9 = –9	
0	–3	–3 – 0 = –3	–3 – (–9) = 6
1	0	0 – (–3) = 3	3 – (–3) = 6
2	9	9 – 0 = 9	9 – 3 = 6

All the second differences are the same, so you can conclude that the derivative of the cubic function is a quadratic function. In the next lesson, you will build on this to assert that the derivative of any cubic function is a quadratic one and you will be able to find the function.

Example

Open "Lesson 4 Activity 3" on your course page at *ilc.org*.

Use the graph $y = 2x^4 - x^2 - 3$ to complete the following table. Then sketch a graph of the derivative and find the equation for that function:

x	$\frac{dy}{dx}$	First difference	Second difference	Third difference
–3				
–2				
–1				
0				
1				
2				
3				

Note that $\frac{dy}{dx}$ is another notation for the derivative.

Solution

x	$\frac{dy}{dx}$	First difference	Second difference	Third difference
–3	–210			
–2	–60	150		
–1	–6	54	–96	
0	0	6	–48	48
1	6	6	0	48
2	60	54	48	48
3	210	150	96	48

The third differences are equal, so you can conclude that the equation of the derivative function is a cubic polynomial. Recall that a cubic polynomial has degree 3 and that a quartic polynomial has degree 4.

What observation can you make about the derivative of a polynomial function? The derivative of a quadratic function is linear and the derivative of a quartic polynomial is cubic. It turns out that the degree of the derivative of a polynomial function is one less than the degree of the polynomial itself.

Support Questions
(do not send in for evaluation)

Use the applet "Tangent Lines and Polynomials" on your course page at *ilc.org*.

13. a) Graph $f(x) = 2x^2 - 3x$ and complete the following table for $f(x)$:

x	Slope of the tangent
−3	
−2	
−1	
0	
1	
2	
3	

b) Use graphing paper to sketch the derivative and state its type (degree).

14. a) Graph $f(x) = x^4 + 3x - 3$ and complete the following table:

x	$\frac{dy}{dx}$	First difference	Second difference	Third difference
−2				
−1				
0				
1				
2				

b) Use graphing paper to sketch a graph of the derivative and state its type (degree).

There are Suggested Answers to Support Questions at the end of this unit.

First Principle Definition of the Derivative

You know from Lesson 2 that the instantaneous rate of change at $x = a$, which is equivalent to the slope of a tangent to a curve where $x = a$, can be found by calculating the following limit:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Since the derivative is the equation that gives the slope of tangents, you can define the derivative of a function f as the function you find by calculating the following:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is referred to as the first principle definition of the derivative.

Example

Determine the derivative of $f(x) = x^2$ using the first principle.

Solution

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{(x+h)^2 - x^2}{h} \\ &= \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{h(2x+h)}{h} \\ &= 2x + h \end{aligned}$$

$$\lim_{h \to 0} 2x + h = 2x$$

The derivative of $f(x) = x^2$ is $2x$.

Example

Determine the derivative of $f(x) = x^3$ using the first principle.

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^3 - x^3}{h}\\ &= \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}\\ &= \frac{3x^2h + 3xh^2 + h^3}{h}\\ &= \frac{h(3x^2 + 3xh + h^2)}{h}\\ &= 3x^2 + 3xh + h^2\end{aligned}$$

As h approaches 0, $\lim\limits_{h \to o}(3x^2 + 3xh + h^2) = 3x^2 + 0 + 0$

$$= 3x^2$$

So, $f'(x) = 3x^2$.

Example

Find the derivative of $f(x) = \sqrt{x+3}$ using the first principle.

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h+3}-\sqrt{x+3}}{h} \\
&= \frac{\sqrt{x+h+3}-\sqrt{x+3}}{h} \times \frac{\sqrt{x+h+3}+\sqrt{x+3}}{\sqrt{x+h+3}+\sqrt{x+3}} \\
&= \frac{(\sqrt{x+h+3}-\sqrt{x+3})(\sqrt{x+h+3}+\sqrt{x+3})}{h(\sqrt{x+h+3}+\sqrt{x+3})} \\
&= \frac{((\sqrt{x+h+3})^2-(\sqrt{x+3})^2)}{h(\sqrt{x+h+3}+\sqrt{x+3})} \\
&= \frac{(x+h+3-x-3)}{h(\sqrt{x+h+3}+\sqrt{x+3})} \\
&= \frac{h}{h(\sqrt{x+h+3}+\sqrt{x+3})} \\
&= \frac{1}{\sqrt{x+h+3}+\sqrt{x+3}}
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{1}{\sqrt{x+h+3}+\sqrt{x+3}} \\
&= \frac{1}{2\sqrt{x+3}}
\end{aligned}$$

Support Questions
(do not send in for evaluation)

15. Calculate $\dfrac{f(x+h)-f(x)}{h}$ and the limit as h goes to 0 for each function:

 a) $f(x) = 2x$

 b) $f(x) = x^2 - 3x + 2$

16. Use the first principle to calculate the derivative of the following functions:

 a) $f(x) = \sqrt{x}$

 b) $f(x) = \dfrac{1}{x-1}$

Conclusion

In this lesson, you used technology to study the derivative of a quadratic, a cubic, and a quartic. You also learned how to calculate the derivative of a polynomial function using the first principle. Using the first principle is time-consuming and tedious. In Lesson 5, you will learn a number of rules to help you determine the derivative without using the first principle.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(16 marks)

8. Calculate the derivative of the following functions using the first principle definition of the derivative.

   a) $f(x) = x^2 - 2x + 1$ **(2 marks)**

   b) $f(x) = x^3 - 3x^2$ **(4 marks)**

   c) $f(x) = \sqrt{2x}$ **(4 marks)**

   d) $f(x) = \dfrac{4}{x+1}$ **(4 marks)**

9. Explain in your own words the process used in this lesson to conclude that the derivative of a cubic function is a quadratic one. **(2 marks)**

Now go on to Lesson 5. Do not submit your coursework to ILC until you have completed Unit 1 (Lessons 1 to 5).

