MCV4U-A

3

Intervals of Increase and Decrease

Introduction

In Lesson 2, you learned about limits and that the instantaneous rate of change of a function at a specific point is the slope of the tangent to the curve at that point. Within the domain of calculus, there is much more information that can be determined from a graph of a function.

If you were shown a graph of a ball being thrown in the air and then falling to the ground, would you be able to point out when the height of the ball is increasing and decreasing? In most cases, it is easy to look at a graph and determine an interval of increase or decrease. In this lesson, you will learn how to calculate these intervals. In addition, you will make further connections between the instantaneous rate of change at a point, the slope of the tangent line at that point, and the way the function behaves.

Estimated Hours for Completing This Lesson	
Increasing and Decreasing Intervals	3.5
Key Questions	1.5

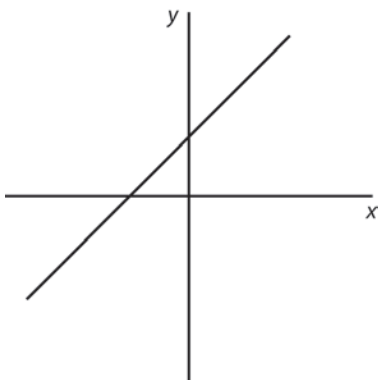
What You Will Learn

After completing this lesson, you will be able to

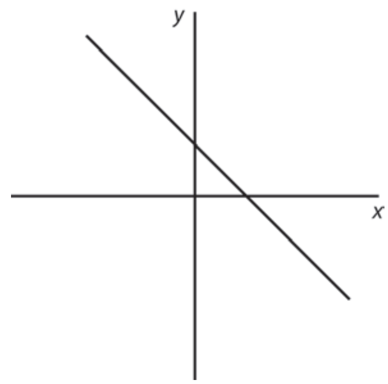
- determine numerically and graphically the intervals over which the instantaneous rate of change is positive, negative, or zero
- determine the behaviour of the function in relation to the instantaneous rate of change at a point

Review of Concepts

Recall that a line with a positive slope goes up from left to right and a line with a negative slope goes down from left to right.



Positive Slope



Negative Slope

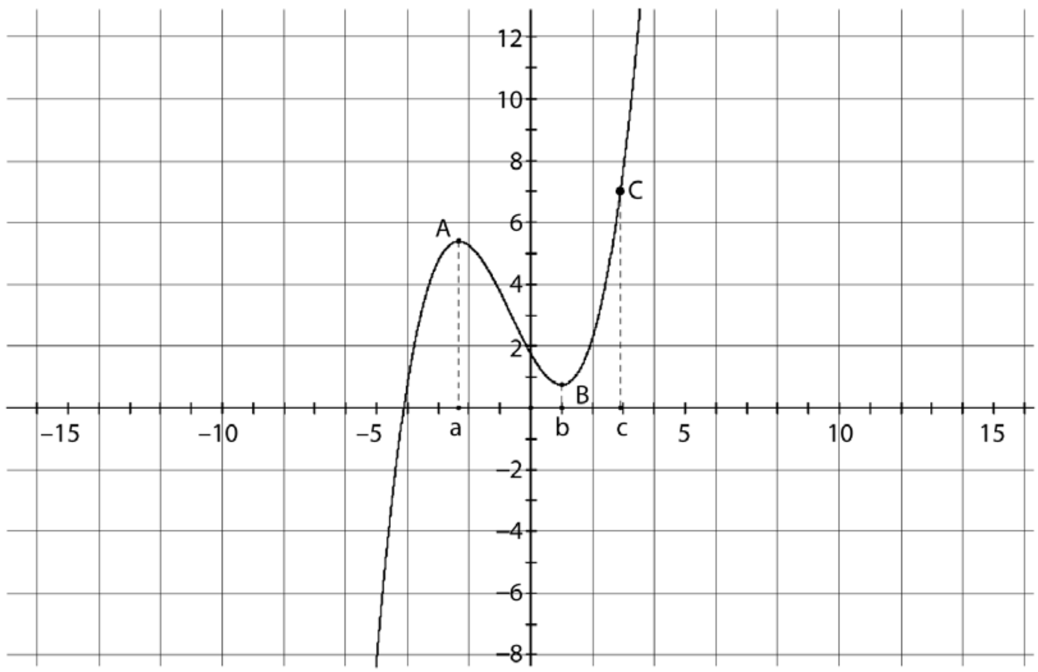
In addition, for this lesson you also need to remember the cube of a sum:

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

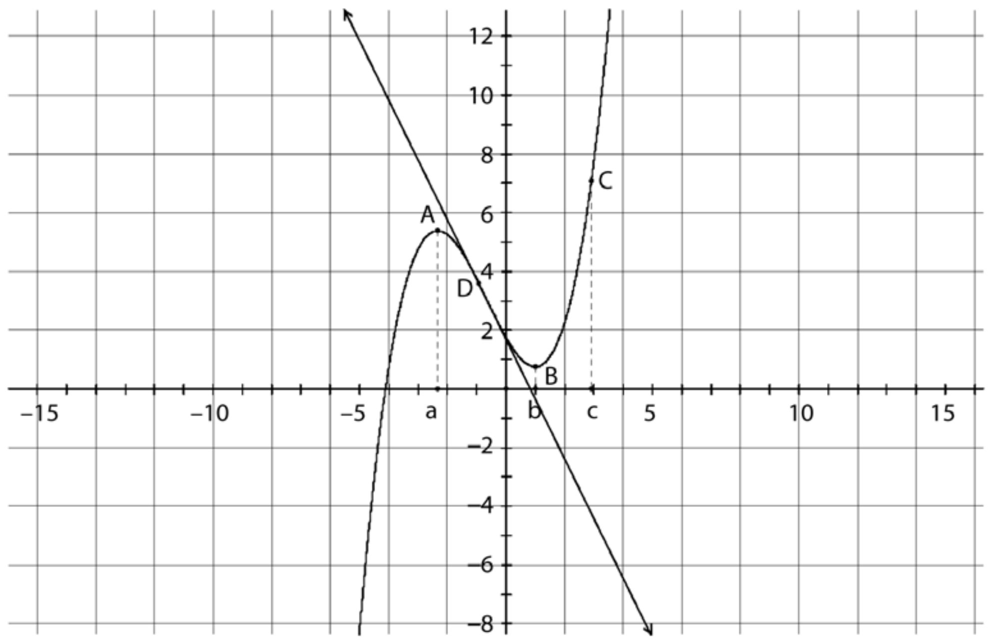
Increasing and Decreasing Intervals

An important property of a function is whether it is increasing (rising) or decreasing (falling) and on what intervals it is increasing or decreasing. Look at the following graph:



The points $A(a, f(a))$, $B(b, f(b))$, and $C(c, f(c))$ are three points on the graph of the function. The function between $x = a$ and $x = b$ is decreasing (it falls) and the function is increasing between $x = b$ and $x = c$ (it rises).

Choose any point between A and B. The instantaneous rate of change at point D is equal to the slope of the tangent to the curve at point D. What can you say about the slope of the tangent at D? Is it positive or negative?



The slope of the tangent at D is negative. Conversely, the slope of the tangent at any point where $x > b$ is positive and therefore the instantaneous rate of change is positive.

What about the instantaneous rate of change at $x = b$? The tangent to the curve is parallel to the x-axis and its slope is zero. A point where the instantaneous rate of change is zero is called a critical point. Observe that the slope of a tangent just less than $x = b$ is negative and the slope at just larger than $x = b$ is positive. The function changes from a decreasing function when x is just less than b to an increasing function when x is just larger than b. Therefore, there is an interval around b where $f(b)$ is the smallest value of the function f for all x in the interval.

You can refer to the $f(b)$ as a local minimum of the function $f(x)$. The $f(b)$ is not the smallest value that the function can have, but it is the smallest for some interval around $x = b$.

Similarly, there is an interval around $x = a$ where $f(a)$ is the largest value of the function for all values of x in that interval. $f(a)$ is called the local maximum of the function.

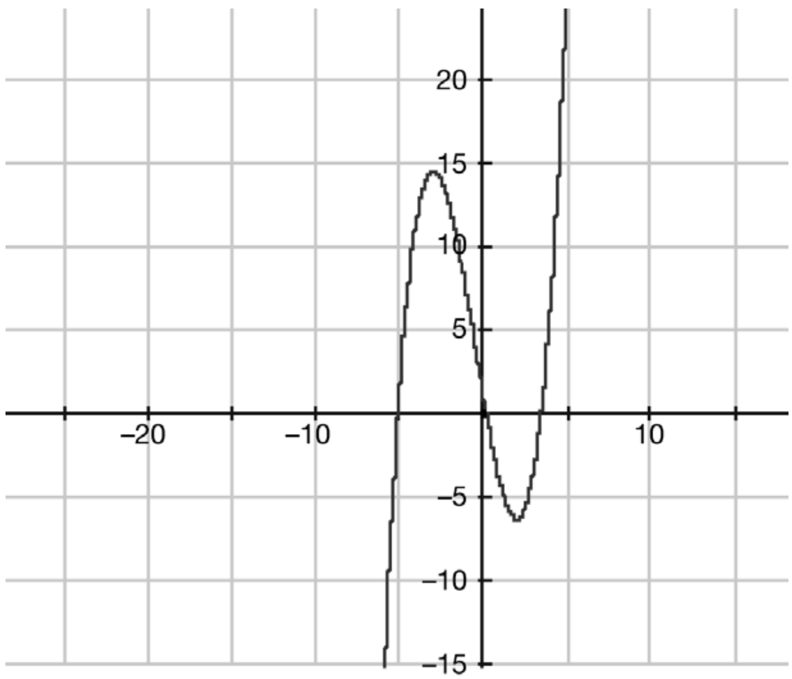
Example

For the following graph:

a) Determine the intervals between which the rate of change is positive and negative.

b) State where the rate of change is zero.

c) Estimate the local maximums and minimums of the function.



Solution

a) The function is increasing when $x < -3$ and $x > 2$, hence its instantaneous rate of change is positive. It is decreasing when $-3 < x < 2$ and hence its instantaneous rate of change is negative.

b) The instantaneous rate of change is 0 at $x = -3$ and $x = 2$.

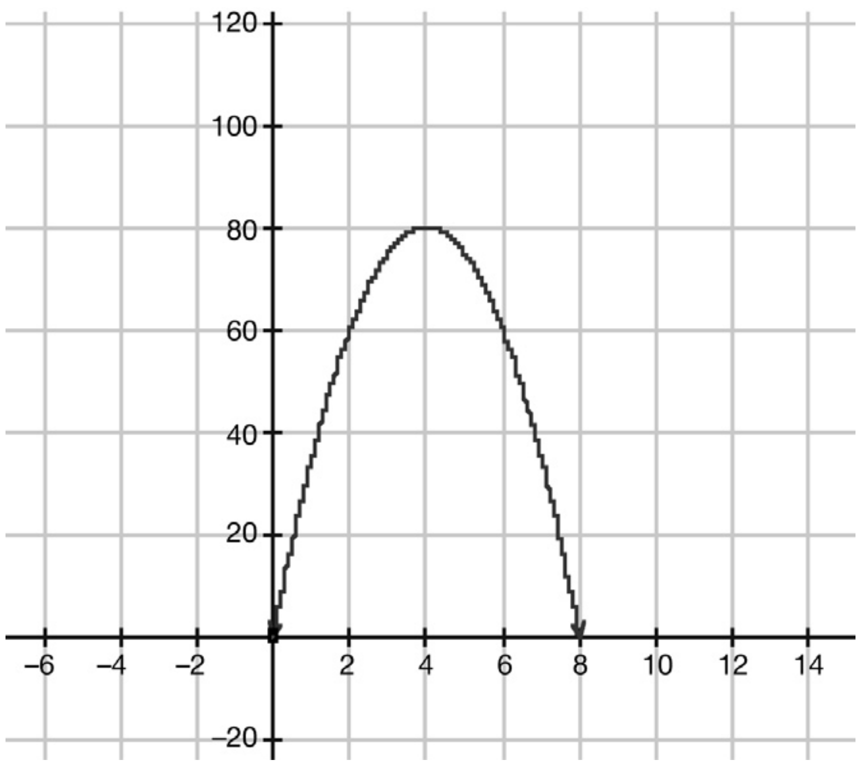
c) $f(-3)$ is a local maximum and it is approximately 14.5.

$f(2)$ is a local minimum and it is approximately -6.3.

Example

A ball is thrown into the air. Its height above the ground (in metres) is modelled by the equation $f(t) = -5t^2 + 40t$, where t is time in seconds. At what time does the ball reach its maximum height? Solve this problem by graphing $f(t)$ to estimate the maximum and confirm your estimation by finding the instantaneous rate of change at that point.

Solution



From the graph, you can estimate the maximum to be 80 m when the time is 4 seconds. If you calculate the instantaneous rate of change of $f(t)$ at $t = 4$, the answer should be zero:

$$\begin{aligned} f(4+h) - f(4) &= -5(4+h)^2 + 40(4+h) + (5)4^2 - 40(4) \\ &= -5(16 + 8h + h^2) + 160 + 40h + 80 - 160 \\ &= -5h^2 - 40h + 40h \\ &= -5h^2 \end{aligned}$$

$$\begin{aligned} \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h \to 0} \frac{-5h^2}{h} \\ &= \lim_{h \to 0}(-5h) \\ &= 0 \end{aligned}$$

The instantaneous rate of change of the height is 0 when $t = 4$. So, $f(4) = 80$ m is the maximum height of the ball.

Support Questions

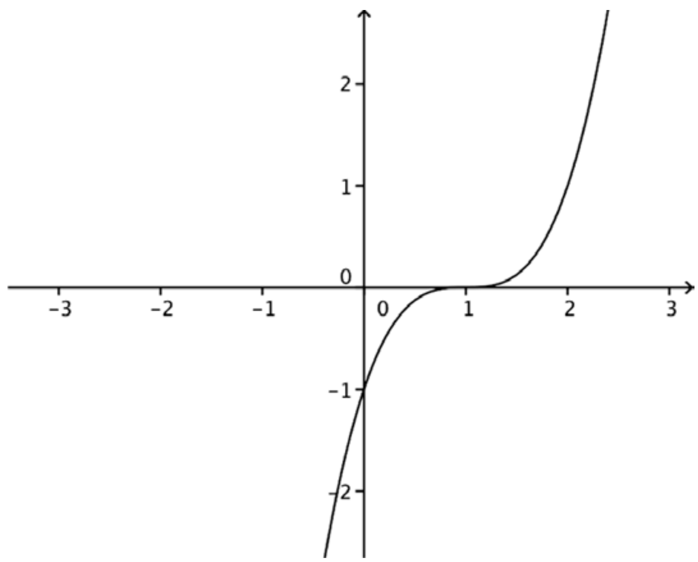
(do not send in for evaluation)

11. For the following graph:

    a) Determine the intervals between which the rate of change is positive and negative.

    b) State where the rate of change is zero.

    c) List the local maximums and minimums of the function.



12. Use the Graphing Applet to estimate the local maximums and local minimums of each function and state the intervals where the function is increasing and decreasing. For each maximum and minimum, show that the instantaneous rate of change is zero.

    a) $f(x) = x^3 + 3x^2 - 24x + 3$

    b) $f(x) = x^3 - 3x^2 + 3x - 1$

Conclusion

In this lesson, you learned about the connection between the instantaneous rate of change of a function and whether the function is increasing or decreasing. While this is helpful, it does not give you a full picture of the behaviour of the function. How do you know what points to check? In Lesson 4, the derivative function will be defined. This will help you analyze functions.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

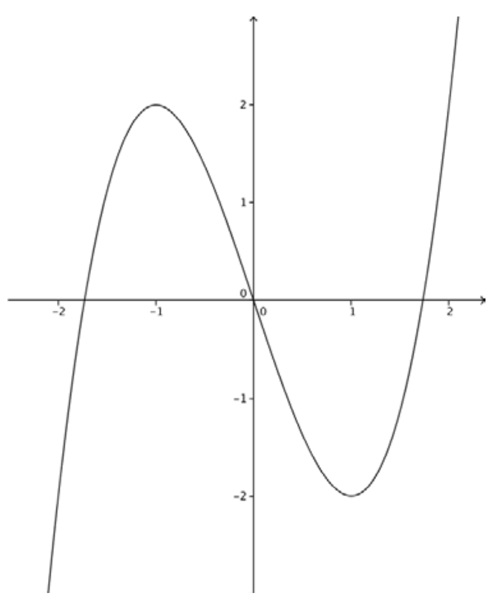
(14 marks)

6. For the following graph:

   a) Determine the intervals between which the rate of change is positive and negative. **(2 marks)**

   b) State where the rate of change is zero. **(1 mark)**

   c) List the local maximums and minimums of the function. **(1 mark)**



7. For the function $f(x) = 2x^3 - 7x^2 + 4x + 1$

   a) Find the instantaneous rate of change at $x = 0$ and $x = 1$. **(6 marks)**

   b) Is the function increasing or decreasing at $x = 0$? How about at $x = 1$? **(2 marks)**

   c) Do you expect a local maximum or a local minimum when $0 < x < 1$? Explain your answer. **(2 marks)**

Now go on to Lesson 4. Do not submit your coursework to ILC until you have completed Unit 1 (Lessons 1 to 5).

