MCV4U-A

2

Limits

Introduction

In this lesson, you will look at the concept of limits and discover how finding limits can help you find instantaneous rates of change.

What exactly is a limit? In everyday life, you may have heard phrases that sound something like this:

“I’ve reached my limit—if this printer fails one more time, I’m buying a new one!”

“I have to limit the use of my flashlight because the batteries are almost dead.”

In both of these statements, the word “limit” is used to describe a boundary or furthest extent possible. The mathematical use of “limit” is the same.

Estimated Hours for Completing This Lesson	
Limits	1.5
Limits and Functions	1.0
Limits and Instantaneous Rates of Change	1.5
Key Questions	1

For this lesson, there is an interactive tutorial on your course page. You may find it helpful during, or at the end of, this lesson to work through the tutorial called “Limits.”

What You Will Learn

After completing this lesson, you will be able to

- understand and find the limit of a sequence
- understand and find the limit of a function as x approaches a value
- find the instantaneous rate of change at a point

Limits

A numeric limit is a number that acts as a boundary—a number you can approach, but not quite get to in your calculations.

Start with an example:

Sequence 1: 1, 2, 3, 4, 5, 6, 7, …

Sequence 2: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots$

Both of these sequences have a very clear pattern to them. If you continued to write values in sequence 1, the numbers would continue to get bigger and bigger. There is no limit to how big they would get. If you continued to write values in sequence 2, on the other hand, there is a limit to how small they would get. This becomes clear if you write the sequence out in decimal form instead of fraction form.

Sequence 2 as a fraction	Sequence 2 as a decimal (accurate to 4 decimal places)
$\frac{1}{1}$	1.0000
$\frac{1}{2}$	0.5000
$\frac{1}{3}$	0.3333
$\frac{1}{4}$	0.2500
$\frac{1}{5}$	0.2000
$\frac{1}{6}$	0.1667
$\frac{1}{7}$	0.1429

You can see just from these few terms that the sequence is getting smaller and smaller in value. (Recall that a term is one of the numbers in a sequence.) If you were to continue to write out the sequence, how small would the values get?

	Sequence 2 as a fraction	Sequence 2 as a decimal (accurate to 3 decimal places)
The tenth term in the sequence …	$\frac{1}{10}$	0.100
The hundredth term in the sequence …	$\frac{1}{100}$	0.010
The thousandth term in the sequence …	$\frac{1}{1000}$	0.001

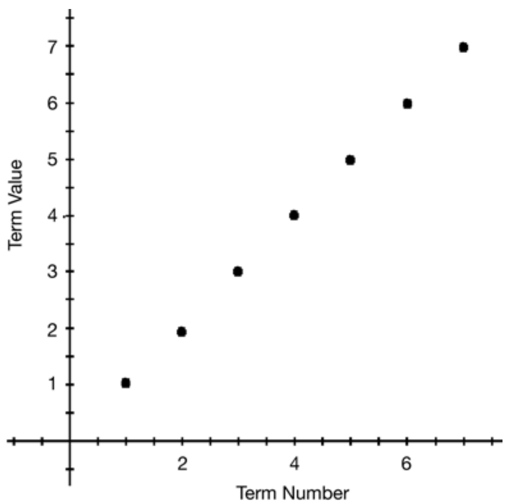
You can see that the sequence does indeed get smaller and smaller. Mathematically, you can say that the value approaches 0, or that 0 is the limiting value of the sequence. You should recognize that as you continue to write terms in the sequence, you will always produce smaller and smaller numbers, but will never get a value of exactly 0.

Visualizing a Limit

With sequences 1 and 2 in the previous section, you used your algebraic or written math skills to examine the concept of a limit. You can also look for limits by graphing:

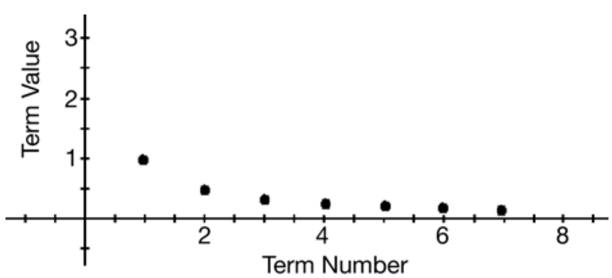
Term Number	Sequence 1 Value	Sequence 2 Value
1	1	$\frac{1}{1}$
2	2	$\frac{1}{2}$
3	3	$\frac{1}{3}$
4	4	$\frac{1}{4}$
5	5	$\frac{1}{5}$
6	6	$\frac{1}{6}$
7	7	$\frac{1}{7}$

In the graph of sequence 1, the points do not appear to be bound by any value. They just continue to increase:



Sequence 1

In the graph of sequence 2, as the term number increases, the points appear to approach zero.



Sequence 2

Asymptotes

On the graph of sequence 2, the points approach zero, but they will never equal zero. The points will continue to get closer and closer, but never actually get there. On the graph, you would say that there is an asymptote at zero. An asymptote is a line (or a curve) that the points on a graph approach but never touch. You probably studied asymptotes in more detail in a previous math course.

Limit Notation

To use proper limit notation, you need to go back and rewrite sequence 2 more mathematically. Write a general statement that says, "If you want the nth term, do the following math." Look again at the sequence:

Sequence 2: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots$

The 1st term is $\frac{1}{1}$.

The 2nd term is $\frac{1}{2}$.

The 3rd term is $\frac{1}{3}$.

And so on.

Therefore, the nth term is $\frac{1}{n}$.

You have discovered, algebraically and graphically, that as you look at later and later terms in the sequence, the value of the term gets closer to, but never quite gets to, zero. Here is how to say the same thing using mathematical symbols:

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

"The limit as n approaches infinity of the expression $\frac{1}{n}$ is 0."

In other words, as n gets bigger and bigger, $\frac{1}{n}$ gets closer and closer to zero.

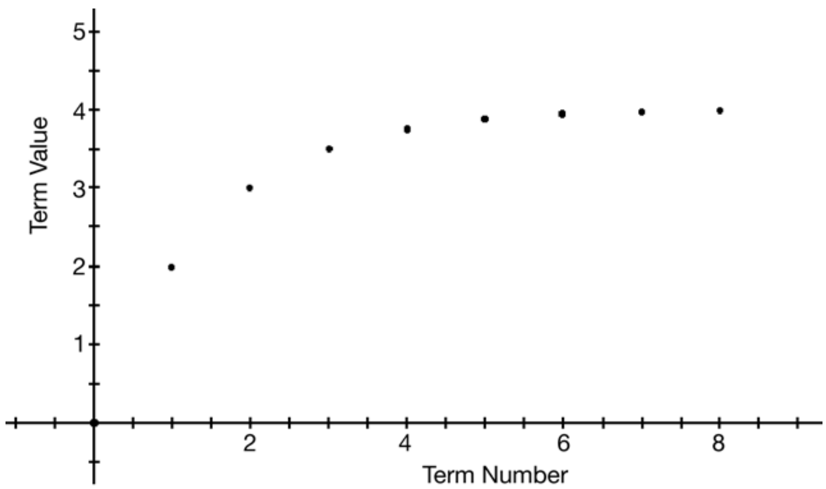
Support Questions

(do not send in for evaluation)

5. Find the limiting value of $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots , \frac{n+1}{n}$.

6. Find the limiting value of $1, \frac{2}{3}, \frac{3}{5}, \ldots , \frac{n}{(2n-1)}$.

7. Find the limiting value of the sequence $f(n)$, shown in the graph:



There are Suggested Answers to Support Questions at the end of this unit.

Limits and Functions

In most of the examples you have looked at so far, you found the limit as n got larger and larger in a particular sequence. You can find limits in functions as well, and not just for values approaching infinity.

Think about the function $f(x) = x^2$. Is there a limit as x approaches infinity? As x takes on values that are larger and larger, the value of x^2 also gets larger and larger. There is no upper boundary limiting the value of x^2.

You can find other limits in this function. For example, evaluate the limit as x approaches 2.

Algebraically:

As you did earlier in the lesson, to evaluate a limit use a table to test values of the function as it gets closer and closer to 2:

x	$f(x)$
1.9	3.6100
1.99	3.9601
1.999	3.9960
1.9999	3.9996

This table clearly shows that as x gets closer to a value of 2, $f(x)$ gets closer to a value of 4. You have not properly found the limit yet. You have found only the left-hand limit—the limiting value as you test values of x that are less than 2. In order to say that a limit exists, you also have to find the right-hand limit—the limiting value as you test x-values that are greater than 2.

x	$f(x)$
2.1	4.4100
2.01	4.0401
2.001	4.0040
2.0001	4.0004

You can see that the right-hand limit is also 4. As the left-hand limit and right-hand limit are the same, you can say that the limit as x approaches 2 is 4. Mathematically, you can write it like this:

$$\lim_{x \to 2} f(x) = 4$$

Right-Hand and Left-Hand Limits

The left-hand limit can also be written using symbols:

$$\lim_{x \to 2^-} f(x) = 4$$

The notation 2^- tells you that this is approaching 2 from below, using values less than 2.

Similarly, the notation 2^+ tells you that you are approaching 2 from above, using values greater than 2. The right-hand limit therefore looks like this:

$$\lim_{x \to 2^+} f(x) = 4$$

As a general rule:

The $\lim\limits_{x \to a} f(x) = b$ only if $\lim\limits_{x \to a^-} f(x) = b$ and $\lim\limits_{x \to a^+} f(x) = b$.

This means the following:

- If the left- and right-hand limits are not the same value, the limit does not exist.
- If domain restrictions do not allow both the left- and right-hand limits to be found, you need only test the limit that exists.
- The actual value at the point is irrelevant. Only the values approaching that point are important.

> **Note:** To be able to solve for some limits, you will need to factor a quadratic relationship. If you do not remember how to do this, consult the Review section at the start of Unit 1.

Here are a number of examples where you can practise evaluating limits.

Examples

Evaluate the following limits:

a) $\lim\limits_{x\to 2} 2x^2 - 4x + 1$

b) $\lim\limits_{x\to 3^+} \dfrac{3x}{x-4}$

c) $\lim\limits_{x\to 2^+} \dfrac{3x+1}{x-2}$

d) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

e) $\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{x-2}$

f) $\lim\limits_{x\to 25} \dfrac{\sqrt{x}-5}{x-25}$

Solutions

a) Since $2x^2 - 4x + 1$ is well defined at $x = 2$ and equal to $2(2)^2 - 4(2) + 1 = 1$, you can conclude that $\lim\limits_{x\to 2} 2x^2 - 4x + 1 = 1$.

b) $\dfrac{3x}{x-4}$ is defined at $x = 3$, so $\lim\limits_{x\to 3^+} \dfrac{3x}{x-4} = \dfrac{3(3)}{(3-4)} = -9$

c) As x approaches 2 from the right, the value of the numerator is $3(2^+) + 1$ or 7 and the denominator approaches $(2^+) - 2$ or 0. You end up with a numerator that approaches 7 and a denominator that approaches 0 from the right. Therefore, $\lim\limits_{x\to 2^+} \dfrac{3x+1}{x-2} = +\infty$

d) In this example, if you substitute 3 into $\frac{x^2-9}{x-3}$, you get the indeterminate form $\frac{0}{0}$. The approach you use is to factor the numerator and divide by the common factor in the numerator and denominator to see if you can evaluate the limit of the reduced function.

Notice that the numerator is a difference of squares and can be written as $(x-3)(x+3)$.

The limit becomes

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)}=\lim_{x\to 3}(x+3)=6$$

e) In this example, if you substitute 2 into $\lim_{x\to 2}\frac{x^2-3x+2}{x-2}$, you get the indeterminate form $\frac{0}{0}$. You should take the approach used when solving (d).

$$\lim_{x\to 2}\frac{x^2-3x+2}{x-2}=\lim_{x\to 2}\frac{(x-2)(x-1)}{x-2}=\lim_{x\to 2}(x-1)=1$$

f) $\lim_{x\to 25}\frac{\sqrt{x}-5}{x-25}$

If you substitute $x = 25$ into $\lim_{x\to 25}\frac{\sqrt{x}-5}{x-25}$, you get the indeterminate form $\frac{0}{0}$. With this type of question, you multiply both numerator and denominator by the conjugate of $\sqrt{x}-5$ which is $\sqrt{x}+5$. Note the conjugate of a sum is the difference and the conjugate of a difference is the sum.

$$\lim_{x\to 25}\frac{\sqrt{x}-5}{x-25}=\lim_{x\to 25}\frac{\sqrt{x}-5}{x-25}\frac{\sqrt{x}+5}{\sqrt{x}+5}=\lim_{x\to 25}\frac{\left(\sqrt{x}-5\right)\left(\sqrt{x}+5\right)}{(x-25)\left(\sqrt{x}+5\right)}.$$

The numerator is of the form $(a-b)(a+b)$, which is equal to a^2-b^2.

So,

$$\lim_{x\to 25}\frac{\left(\sqrt{x}-5\right)\left(\sqrt{x}+5\right)}{(x-25)\left(\sqrt{x}+5\right)}=\lim_{x\to 25}\frac{\left(\sqrt{x}\right)^2-5^2}{(x-25)\left(\sqrt{x}+5\right)}=\lim_{x\to 25}\frac{(x-25)}{(x-25)\left(\sqrt{x}+5\right)}$$

Divide out $x - 25$ to get $\lim\limits_{x\to 25}\dfrac{1}{\left(\sqrt{x}+5\right)}$.

Substitute 25 in the function $\dfrac{1}{\left(\sqrt{x}+5\right)}$ to get $\dfrac{1}{\sqrt{25}+5}=\dfrac{1}{10}$.

Therefore, $\lim\limits_{x\to 25}\dfrac{\sqrt{x}-5}{x-25}=\dfrac{1}{10}$.

Support Question
(do not send in for evaluation)

8. Evaluate the following limits:

a) $\lim\limits_{x\to 2} 3x^2 - 4$

b) $\lim\limits_{x\to 2}\dfrac{2}{x-3}$

c) $\lim\limits_{x\to -1}\dfrac{x^2-3x-4}{x^2-1}$

d) $\lim\limits_{x\to 0}\dfrac{\sqrt{4-x}-2}{x}$

Limits and Instantaneous Rates of Change

Now that you have a better understanding of limits in general, they can be used to find instantaneous rates of change.

Recall from Lesson 1 that you can calculate the *average* rate of change between two points by using the formula $\frac{f(b)-f(a)}{b-a}$. You also learned that you can estimate the *instantaneous* rate of change at point $(a, f(a))$ by using the same formula and selecting a value for b that is very close to the value of a.

For example: To estimate the instantaneous rate of change at $t = 5$ seconds in the formula $N(t) = t^2 + 4t + 1$, you might use $a = 5$ and $b = 5.01$. Another student doing the same question may have used $b = 5.001$. Yet another might have used $b = 5.0001$. All selections will yield the correct answer, but the closer b is to a, the better your estimate for instantaneous rate of change.

To estimate instantaneous rate of change for a function at point $(a, f(a))$, select a second point in the form $(a + h, f(a + h))$, where h is a very small value. You can then estimate the rate of change at point a using the formula $\frac{f(a+h)-f(a)}{(a+h)-a}$.

Now use this revised formula in an example.

Example

Estimate the instantaneous rate of change at the point where $x = 2$ in the function $f(x) = x^3 - 2x + 1$.

Solution

First you need to calculate $f(2)$:

$$\begin{aligned} f(2) &= (2)^3 - 2(2) + 1 \\ &= 8 - 4 + 1 \\ &= 5 \end{aligned}$$

In the following table, you need to find the value of $\dfrac{f(a+h)-f(a)}{(a+h)-a}$ for decreasing values of h to estimate the instantaneous rate of change:

Value of h	$a + h$	$f(a + h)$	$\dfrac{f(a+h)-f(a)}{(a+h)-a}$
0.1	2.1	$f(2.1) = (2.1)^3 - 2(2.1) + 1$ $= 9.261 - 4.2 + 1$ $= 6.061$	$\dfrac{f(a+h)-f(a)}{(a+h)-a} = \dfrac{6.061-5}{(2.1)-2}$ $= \dfrac{1.061}{0.1}$ $= 10.61$
0.01	2.01	5.100601	$\dfrac{f(a+h)-f(a)}{(a+h)-a} = \dfrac{5.100601-5}{(2.01)-2}$ $= \dfrac{0.100601}{0.01}$ $= 10.0601$
0.001	2.001	5.010006001	$\dfrac{f(a+h)-f(a)}{(a+h)-a} = \dfrac{5.010006001-5}{(2.001)-2}$ $= \dfrac{0.010006001}{0.001}$ $= 10.006001$

You could continue to select smaller and smaller values for h, but how small is small enough? What you really want is to let h get closer and closer to zero, which can be expressed as the following limit statement. Simplify the denominator of the rate of change $(a + h) - a = h$ to come up with the following definition:

> **Instantaneous rate of change** at the point $(a, f(a))$ is $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

Example

Now use this formula to determine the instantaneous rate of change where $x = 2$ in the function $f(x) = x^3 - 2x + 1$.

Solution

From your work in the previous example, you might be able to estimate the rate of change to be 10. Now you will use the new limit definition to calculate the exact rate of change.

First, calculate $f(2)$ and $f(2 + h)$:

$f(2) = 5$

$$\begin{aligned} f(2+h) &= (2+h)^3 - 2(2+h) + 1 \\ &= (8 + 12h + 6h^2 + h^3) - 2(2+h) + 1 \\ &= 8 + 12h + 6h^2 + h^3 - 4 - 2h + 1 \\ &= 5 + 10h + 6h^2 + h^3 \end{aligned}$$

Apply these values to the formula:

Instantaneous rate of change at the point (2, 5) =

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$\begin{aligned} &= \lim_{h\to 0}\frac{(5+10h+6h^2+h^3)-5}{h} \\ &= \lim_{h\to 0}\frac{10h+6h^2+h^3}{h} \\ &= \lim_{h\to 0}\frac{h(10+6h+h^2)}{h} \\ &= \lim_{h\to 0} 10+6h+h^2 \end{aligned}$$

At this stage, rather than estimating the value of this limit by using values of h such as 0.0001, you can actually let $h = 0$ because of the algebraic simplification you were able to do. In other words, you can find the "slope of the secant" between $(2, f(2))$ and $(2 + 0, f(2 + 0))$, which is really the slope at a single point.

$$\begin{aligned} \lim_{h\to 0} 10+6h+h^2 &= 10 + 6(0) + (0)^2 \\ &= 10 \end{aligned}$$

Support Questions
(do not send in for evaluation)

9. For the function $f(x) = 3x^2 - 2x + 1$, determine the value of the following:
   a) The slope of the secant between $x = 3$ and $x = 3.5$
   b) The average rate of change between $x = 3$ and $x = 3.5$
   c) The $\lim\limits_{h \to 0}$ of the secant slope between $f(3 + h)$ and $f(3)$
   d) The instantaneous rate of change at $x = 3$

10. A cannon fired from a cliff follows a path modelled by the equation $H(t) = -5t^2 + 50t + 80$, where H is the height above ground in metres and t is the time in seconds since the cannon was fired. What is the instantaneous velocity at $t = 1$ second?

Conclusion

In this lesson, you learned about limits and how limits can help you find the instantaneous rate of change of a function at a specific point. In Lesson 3, you will learn how finding the instantaneous rate of change of a function at a specific point helps you learn about the behaviour of the function itself.

Key Questions

Save your answers to the Key Questions. When you have completed the unit, submit them to ILC for marking.

(13 marks)

4. a) Describe in your own words how you would determine the instantaneous rate of change of a function using the methods discussed in this lesson. **(3 marks)**

   b) Evaluate the following limits:

   i) $\lim_{x \to 0} \frac{x-3}{2x^2-5}$ **(1 mark)**

   ii) $\lim_{x \to 2} \frac{2x^2-7x+6}{x-2}$ **(2 marks)**

5. a) A ball is thrown in the air. Its height from the ground in metres after t seconds is modelled by $h(t) = -5t^2 + 20t + 1$. What is the instantaneous velocity of the ball at $t = 2$ seconds? **(4 marks)**

   b) A particle's motion is described by the equation $d = t^2 - 8t + 15$ where d and t are measured in metres and seconds. Show that the particle is at rest when $t = 4$. **(3 marks)**

Now go on to Lesson 3. Do not submit your coursework to ILC until you have completed Unit 1 (Lessons 1 to 5).

