Oct 1

Lecture 22

Let $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

a) Find B^2
b) Find B^{-1}
c) What are the eigenvalues of B
d) Find a matrix A such that $A^2 = B$.

a) $\begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 25 \end{bmatrix}$

b) $\left[\begin{array}{ccc|ccc} 4 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$

c) $\lambda = 5, 4, 1$ $\qquad \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(5-\lambda)$

d) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}$

- $e^B = \begin{bmatrix} e^4 & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^5 \end{bmatrix}$ (later...)

Diagonalization

We "diagonalize" matricies by using eigenvalues and eigenvectors

ex $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ · we first find eigenvalues and eigenvectors

$$C_A(\lambda) = \det(A - \lambda I_2)$$
$$= \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1$$
$$= \lambda^2 - 6\lambda + 9 - 1$$
$$= \lambda^2 - 6\lambda + 8$$

$= (\lambda-2)(\lambda-4)$

- $\lambda_1 = 2 \quad \lambda_2 = 4$ are the eigenvalues.

$E_2 = \ker(A - 2I_2)$
$E_4 = \ker(A - 4I_2)$

$E_2: \ [A - 2I_2 \mid 0] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = -s \\ x_2 = s \end{array}$

$E_2 = \text{span}\{(-1,1)\}$

$E_4: \ [A - 4I_2 \mid 0] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{array}{l} x_1 = +s \\ x_2 = s \end{array}$

$E_4 = \text{span}\{(1,1)\}$

$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ • $\{v_1, v_2\}$ is a basis for $\mathbb{R}^2$ consisting of eigenvectors of A.
$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

If we let $P = [v_1 \ v_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, P is then invertible

$AP = A\,[v_1 \ v_2]$
$= [Av_1 \ Av_2]$
$= [2v_1 \ 4v_2]$

$Av_1 = 2v_1$ "because $v_1 \in E_2$"
$Av_2 = 4v_2$ "because $v_2 \in E_4$"

Let $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ • Eigenvalues of A.

$PD = [v_1 \ v_2] \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$= [v_1 \ v_2] \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

$= [2v_1 \ 4v_2]$

$PD = AP$
$P^{-1}PD = A\underbrace{PP^{-1}}_{I_n}$

$\boxed{A = PDP^{-1}}$

- Suppose you want to know A^{201}

$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{201}$

$A^{201} = (PDP^{-1})^{201}$

$= \underbrace{PDP^{-1}PDP^{-1}}_{I_2 \quad I_2} \cdots \underbrace{}_{I_2} PDP^{-1}$

$= \underbrace{PDD \ldots D}_{201 \text{ times}} P^{-1}$

$= PD^{201}P^{-1}$

$= P \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}^{201} P^{-1}$

$= A^{201} = P \begin{bmatrix} 2^{201} & 0 \\ 0 & 4^{201} \end{bmatrix} P^{-1}$

<u>Definition</u>: Any $n \times n$ matrix is called <u>diagonlizable</u> if there exists a diagonal matrix D, and an invertible matrix P such that $A = PDP^{-1}$ or $D = (P^{-1}AP)$

<u>Facts)</u>:

1) a matrix is digonalizeable $\Leftrightarrow$ there is a basis $\{v_1, \ldots v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A. ie, there exist $\lambda_i \in \mathbb{R}$, $i = 1 \ldots n$ such that $Av_i = \lambda v_i$

- In this case we find P and D as follows:

$P = [v_1 \; v_2 \ldots v_n]$

$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ \vdots & \lambda_2 & \cdots & & 0 \\ & \vdots & \ddots & & \\ 0 & \vdots & & & \lambda_n \end{bmatrix}$

where $Av_1 = \lambda_1 v_1$ $\quad \boxed{Av_n = \lambda_n v_n}$

$Av_2 = \lambda_2 v_2$

ie. the eigen vecter has to line up with it's eigenvalue

Facts):

2) A is diagnolizable $\Longleftrightarrow$ $\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) + \dots \dim(E_{\lambda_n}) = n$

3) Special case: A has n different eigenvalues

Example: $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ is diaganalizable with $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

Example 2: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is not diagonizable because:

$$C_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$
$$= (2-\lambda)(2-\lambda)$$
$$\therefore \lambda = 2$$

$$E_2 = [A - 2I_2 | 0]$$
$$= \left[\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right] \quad \begin{matrix} x_1 = s \\ x_2 = 0 \end{matrix}$$

$E_2 = \text{span}\{(1,0)\}$

- We can't find a basis of $\mathbb{R}^2$ consisting of eigenvectors because all of the eignvecters of A are inside of this set $= \text{span}\{(1,0)\}$

$\therefore$ A is not diagonlizable.

Ex: $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad C_A(\lambda) = \begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2$

$\therefore$ $\lambda = 3$ is the only eigenvalue.

E_3 $[A-3I_2 \mid 0] = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x_1 = s$, $x_2 = t$

$E_3 = \text{span}\{\underbrace{(1,0),(0,1)}_{2\text{ vectors}}\}$ • So this is a basis of $\mathbb{R}^2$, hence this can be diagonalizable.

$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$A = PDP^{-1}$

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓ Both sides are equal

Note: every diagonal matrix, A is diagonalizable with $D = A$, $P = I$

Ex • $I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$C_I(\lambda) = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$

$C_I(\lambda) = \lambda^2 + 1$ has no real roots, so it is not diagonalizable over $\mathbb{R}$. But does over $\mathbb{Z}$.

$\lambda^2 + 1 = (\lambda + i)(\lambda - i)$

$\lambda = \pm i$ $\therefore$ I is diagonalizable over $\mathbb{Z}$.

Warning: Diagonalization and Invertibility are not correlated. Even though P needs to be invertible A doesn't have to be. A can be diagonalizable but not invertible and viceversa.