Leo Papandreou
#1107607
MAST 330/2/AA 2010
Assignment 7
2010-10-26

Boyce & DiPrima 3.1.7, 3.1.13, 3.3.11, 3.3.18, 3.3.22, 3.4.4, 3.4.14, 3.4.23

3.1.7

$$y'' - 9y' + 9 = 0$$

$$y = e^{rt} \implies r^2 - 9r + 9 = 0 \implies r_1 = \frac{9+3\sqrt{5}}{2},\ r_2 = \frac{9-3\sqrt{5}}{2}$$

Distinct real roots $r_1, r_2 \implies y = C_1e^{r_1t} + C_2e^{r_2t}$. Thus

$$y = C_1 e^{\frac{9+3\sqrt{5}}{2}t} + C_2 e^{\frac{9-3\sqrt{5}}{2}t}$$

■

3.1.13

$$y'' + 5y' + 3y = 0 \quad y(0) = 1,\ y'(0) = 0$$

Find the solution of the given IVP, sketch its graph and describe its behavior for increasing t.

$$y = e^{rt} \implies r^2 + 5r + 3 = 0 \implies r_1 = \frac{-5+\sqrt{13}}{2},\ r_2 = \frac{-5-\sqrt{13}}{2}$$

Distinct real roots $r_1, r_2 \implies y = C_1e^{r_1t} + C_2e^{r_2t}$. Thus

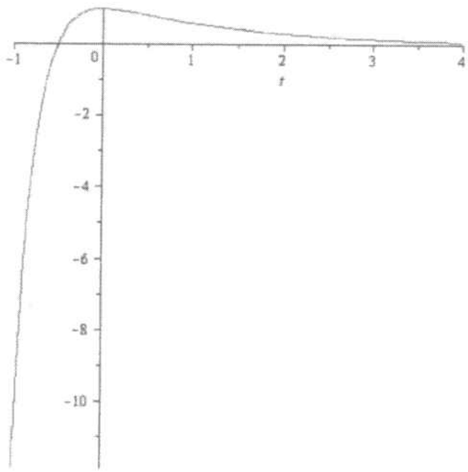
$$y = C_1 e^{\frac{-5+\sqrt{13}}{2}t} + C_2 e^{\frac{-5-\sqrt{13}}{2}t}$$

$$y' = \left(\frac{-5+\sqrt{13}}{2}\right) C_1 e^{\frac{-5+\sqrt{13}}{2}t} + \left(\frac{-5-\sqrt{13}}{2}\right) C_2 e^{\frac{-5-\sqrt{13}}{2}t}$$

$$\begin{aligned} y(0) &= 1 = C_1 + C_2 \\ y'(0) &= 0 = \left(\frac{-5+\sqrt{13}}{2}\right) C_1 + \left(\frac{-5-\sqrt{13}}{2}\right) C_2 \end{aligned}$$

$$C_1 = -\frac{\left(\frac{-5-\sqrt{13}}{2}\right)}{\left(\frac{-5+\sqrt{13}}{2}\right)} C_2 = \frac{-5-\sqrt{13}}{-5+\sqrt{13}} (C_1 - 1) \implies C_1 = \frac{1}{26}\left(13 + 5\sqrt{13}\right),\ C_2 = \frac{1}{26}\left(13 - 5\sqrt{13}\right)$$

$$\therefore y = \frac{1}{26}\left(13 + 5\sqrt{13}\right) e^{\frac{-5+\sqrt{13}}{2}t} + \frac{1}{26}\left(13 - 5\sqrt{13}\right) e^{\frac{-5-\sqrt{13}}{2}t}$$



Since $5 > \sqrt{13}$, y has the form $Ae^{-at} + (1 - A)\, e^{-bt}$ where both a and b are > 0. Therefore

$$\lim_{t\to\infty} y = A \lim_{t\to\infty} \frac{1}{e^{at}} + (1 - A) \lim_{t\to\infty} \frac{1}{e^{bt}} = 0 + 0 = 0$$

■

3.3.11

$$y'' + 6y' + 13y = 0$$

$$y = e^{rt} \implies r^2 + 6r + 13 = 0 \implies r = -\frac{6}{2\,(1)} \pm \frac{\sqrt{6^2 - 4(1)\,(13)}}{2\,(1)} \implies r_1 = -3 + 2i,\ r_2 = -3 - 2i$$

Conjugate complex roots $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu \implies y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$. Thus

$$y = C_1 e^{-3t} \cos 2t + C_2 e^{-3t} \sin 2t$$

■

3.3.18

$$y'' + 4y' + 5y = 0, \quad y\,(0) = 1, \quad y'\,(0) = 0$$

Find the solution of the given IVP, sketch its graph and describe its behavior for increasing t.

$$y = e^{rt} \implies r^2 + 4r + 5 = 0 \implies r = -\frac{4}{2\,(1)} \pm \frac{\sqrt{4^2 - 4(1)(5)}}{2\,(1)} \implies r_1 = -2 + i,\ r_2 = -2 - i$$

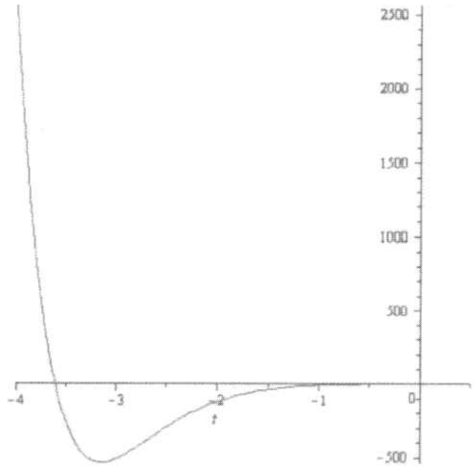
Conjugate complex roots $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu \implies y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$. Thus

$$y = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t$$

$$\begin{aligned} y' &= -2C_1e^{-2t}\cos t - C_1e^{-2t}\sin t - 2C_2e^{-2t}\sin t + C_2e^{-2t}\cos t \\ &= e^{-2t}\left[(-2C_1 + C_2)\cos t + (-C_1 - 2C_2)\sin t\right] \end{aligned}$$

$$\begin{aligned} y(0) &= 1 = C_1e^{-2t}\cos t + C_2e^{-2t}\sin t \,|_{t=0} = C_1 \\ y'(0) &= 0 = e^{-2t}\left[(-2C_1 + C_2)\cos t + (-C_1 - 2C_2)\sin t\right]|_{t=0} = -2C_1 + C_2 \end{aligned}$$

$$\therefore \begin{array}{l} C_1 = 1 \\ C_2 = 2 \end{array} \implies y = e^{-2t}\cos t + 2e^{-2t}\sin t$$



The behavior of y as $t \to \infty$ is

$$\lim_{t\to\infty}\left[e^{-2t}\cos t + 2e^{-2t}\sin t\right] = \lim_{t\to\infty}\frac{\cos t}{e^{2t}} + 2\lim_{t\to\infty}\frac{\sin t}{e^{2t}} = 0$$

which follows (roughly) from the well known result that $\lim_{x\to\infty}\dfrac{oscillating\,(x)}{x} = 0$:

$$\lim_{t\to\infty}\frac{oscillating_1\,(t)}{e^{2t}} = \lim_{x\to\infty}\frac{oscillating_1\left(\frac{1}{2}\ln x\right)}{x} = \lim_{x\to\infty}\frac{oscillating_2\,(x)}{x} = 0$$

■

3.3.22 ✓

$$y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$$

Find the solution of the given IVP, sketch its graph and describe its behavior for increasing t.

$$y = e^{rx} \implies r^2 + 2r + 2 = 0 \implies r = -\frac{2}{2\,(1)} \pm \frac{\sqrt{2^2 - 4(1)(2)}}{2\,(1)} \implies r_1 = -1 + i,\ r_2 = -1 - i$$

Conjugate complex roots $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu \implies y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$. Thus

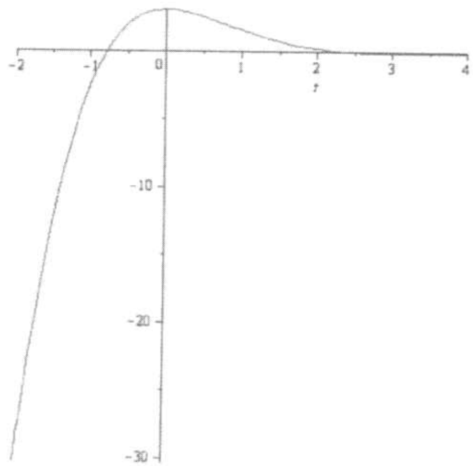
$$y = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t$$

$$y' = -e^{t} \left(C_1 \cos t + C_1 \sin t + C_2 \sin t - C_2 \cos t \right)$$

$$\begin{aligned}
y\left(\frac{\pi}{4}\right) &= 2 = & C_1 e^{-t} \cos t + C_2 e^{-t} \sin t \,\Big|_{t=\frac{\pi}{4}} &= \frac{1}{\sqrt{2} e^{\pi/4}} \left(C_1 + C_2 \right) \\
y'\left(\frac{\pi}{4}\right) &= -2 = & -e^{t} \left(C_1 \cos t + C_1 \sin t + C_2 \sin t - C_2 \cos t \right) \Big|_{t=\frac{\pi}{4}} &= -\frac{1}{\sqrt{2} e^{\pi/4}} 2C_1
\end{aligned}$$

$$\begin{aligned}
2\sqrt{2} e^{\pi/4} &= C_1 + C_2 \\
\sqrt{2} e^{\pi/4} &= C_1
\end{aligned} \implies C_1 = C_2 = \sqrt{2} e^{\pi/4}$$

$$\begin{aligned}
\therefore y &= \sqrt{2} e^{\pi/4} e^{-t} \left(\cos t + \sin t \right) \\
&= \sqrt{2} e^{-\left(t - \frac{\pi}{4}\right)} \left(\cos t + \sin t \right)
\end{aligned}$$



$$\lim_{t \to \infty} \sqrt{2} \frac{1}{e^{t - \pi/4}} \left(\sin t + \cos t \right) \sim k \lim_{t \to \infty} \frac{oscillating\,(t)}{t} = 0$$

■

3.4.4 ✓

$$4y'' + 12y' + 9y = 0$$

$$y = e^{rt} \implies 4r^2 + 12r + 9 = (2r+3)^2 = 0 \implies r_1 = r_2 = -\frac{3}{2} = r$$

Repeated real roots $\implies y = C_1e^{rt} + C_2te^{rt}$. Thus

$$y = C_1e^{-3t/2} + C_2te^{-3t/2}$$

■

3.4.14 ✓

$$y'' + 4y + 4 = 0 \quad y(-1) = 2, \quad y'(-1) = 1$$

Find the solution of the given IVP, sketch its graph and describe its behavior for increasing t.

$$y = e^{rx} \implies r^2 + 4r + 4 = (r+2)^2 = 0 \implies r_1 = r_2 = -2 = r$$

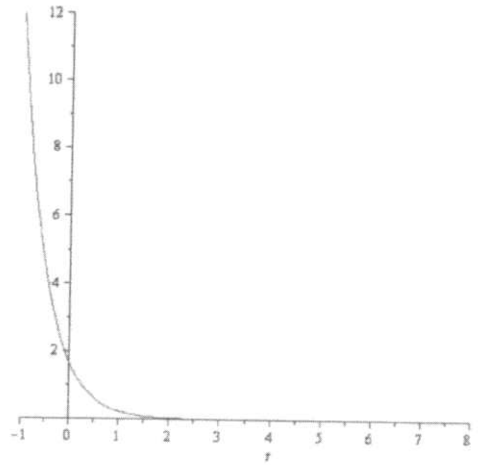
Repeated real roots $\implies y = C_1e^{rt} + C_2te^{rt}$. Thus

$$y = C_1e^{-2t} + C_2te^{-2t}$$

$$\begin{aligned} y(-1) &= C_1e^{-2t} + C_2te^{-2t}\,|_{t=-1} &= C_1e^2 - C_2e^2 &= 2 \\ y'(-1) &= -2C_1e^{-2t} + \left(-2C_2te^{-2t} + C_2e^{-2t}\right)|_{t=-1} &= -2C_1e^2 + 3C_2e^2 &= 1 \end{aligned}$$

$$\begin{array}{rcrcl} C_1 & - & C_2 & = & 2e^{-2} \\ -2C_1 & + & 3C_2 & = & e^{-2} \end{array} \implies C_2 = 5e^{-2},\ C_1 = 7e^{-2}$$

$$\begin{aligned} \therefore y &= 7e^{-2}e^{-2t} + 5te^{-2}e^{-2t} \\ &= 7e^{-2(1+t)} + 5te^{-2(1+t)} \end{aligned}$$



$$\begin{aligned}\lim_{t\to\infty}\left[7e^{-2(1+t)}+5te^{-2(1+t)}\right] &= 7\lim_{t\to\infty}\frac{1}{e^{2(1+t)}}+5\lim_{t\to\infty}\frac{t}{e^{2(1+t)}} \\ &= 0+5\lim_{t\to\infty}\frac{1}{2e^{2(1+t)}} \qquad \text{(L'Hpital's rule)} \\ &= 0\end{aligned}$$

■

3.4.23 Use the method of reduction of order to find a second solution of

$$t^2y''-4ty'+6y=0, \quad t>0; \quad y_1(t)=t^2$$

Let $y_2=v(t)\,y_1(t)$. Then

$$\begin{aligned}y_2 &= vt^2 \\ y_2' &= v't^2+2tv \\ y_2'' &= v''t^2+4tv'+2v\end{aligned}$$

Substitute y_2 and its derivatives into original DE,

$$t^2\left(v''t^2+4tv'+2v\right)-4t\left(v't^2+2tv\right)+6\left(vt^2\right)=0$$

and simplify.

$$t^4v''=0$$

Then, because $t>0$,

$$t^4v'' = 0 \implies v''=0 \;\therefore\; v'=c \text{ and } v=ct+k$$

Since $y_2=vy_1$,

$$\begin{aligned}y_2 &= y_1(ct+k) \\ &= cty_1+ky_1\end{aligned}$$

which is a linear combination of y_1 and ty_1, so only ty_1 provides a new solution,

$$y_2=ty_1=t^3$$

and our fundamental set is

$$\{y_1,y_2\}=\{y_1,ty_1\}=\{t^2,t^3\}$$

Therefore the general solution is

$$y=C_1t^2+C_2t^3$$

■